MEDHA AI

APPSC PAPER I: GENERAL STUDIES & MENTAL ABILITY · QUANTITATIVE APTITUDE AND REASONING

Number system and simplification

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1. Number types and divisibility rules
2. Remainders, unit digits and factors
3. HCF and LCM: methods and word problems
4. Simplification and approximation: BODMAS, surds, indices and roots
5. Number puzzles: digits, sums and differences
6. Decimal fractions and recurring decimals: conversion and comparison
7. Squares, square roots, cubes and cube roots: fast methods and estimation
8. Surds and indices: laws, rationalising and comparing surds
9. Factorials: trailing zeros and the highest power of a prime in n!
10. Counting factors: number and sum of factors, perfect squares and co-primes
11. Remainder theorems: cyclicity, Fermat, Wilson and negative remainders
12. Problems on numbers: turning word statements into equations
13. Speed maths: multiplication, squaring and percentage shortcuts
14. Place value, Roman numerals and counting digits in a range

1 Number types and divisibility rules

Know which family a number belongs to and test divisibility from its digits, so that "find the missing digit", "nearest multiple" and "how many numbers between" questions take seconds.

Core facts

- **Natural numbers:** 1, 2, 3, … **Whole numbers:** 0, 1, 2, … **Integers:** …, −2, −1, 0, 1, 2, …
- **Rational:** can be written p/q with q ≠ 0; its decimal **terminates or recurs**. **Irrational:** non-terminating, non-recurring (√2, √3, π).
- **Prime:** exactly two factors (1 and itself). **1 is neither prime nor composite.**
- **2** is the only even prime; **3** is the first odd prime; **4** is the smallest composite; **9** is the smallest odd composite.
- There are **25 primes below 100** and **15 primes below 50**.
- **Co-prime:** two numbers whose HCF is 1 (8 and 15). They need not be prime. **Twin primes** differ by 2 (11, 13).
- **Prime test:** check divisibility only by primes up to **√n**.
- **Parity:** odd ± odd = even; even ± odd = odd; odd × odd = odd; any product with an even factor is even.
- **Face value** of a digit is the digit itself. **Place value** = digit × its place.
- **Divisibility tests:**

By	Rule
2, 5, 10	last digit even / 0 or 5 / 0
3, 9	digit sum divisible by 3 / 9
4, 25	last two digits divisible by 4 / 25
8, 16	last three / last four digits divisible by 8 / 16
11	(sum of alternate digits) − (sum of the others) is 0 or a multiple of 11
6, 12, 15, 18, 24	2 and 3 / 3 and 4 / 3 and 5 / 2 and 9 / 3 and 8

- Combine two tests only when the two divisors are **co-prime** (24 = 3 × 8, not 4 × 6).

Example 1. Is 247 a prime number? *Solution.* √247 is about 15.7, so test the primes 2, 3, 5, 7, 11, 13. 247 = 13 × 19. **Answer: No, it is composite.**

Example 2. In 46,218, what is the place value of 6 minus its face value? *Solution.* 6 is in the thousands place: place value 6,000, face value 6. **Answer: 5,994**

Example 3. Is 73,416 divisible by 8? *Solution.* Last three digits 416 = 8 × 52. **Answer: Yes.**

Explained

Type 1: one missing digit, one divisor. Write the test as an equation in the digit.

Example 4. 58x3 is divisible by 11. Find x. (a) 2 (b) 4 (c) 6 (d) 8 *Solution.* Alternate sums: (5 + x) − (8 + 3) = x − 6. It must be 0 or a multiple of 11, so x = 6. Check: 5,863 = 11 × 533. **Answer: (c) 6**

Type 2: one digit, two divisors. Apply the stricter test first, then confirm with the other.

Example 5. Replace * in *74381 so the number is divisible by both 3 and 11. (a) 1 (b) 4 (c) 7 (d) 9* Solution.* For 11: (7 + 4 + 8) − (* + 3 + 1) = 15 − , *so* = 4. For 3: digit sum 23 + 4 = 27, divisible by 3. **Answer: (b) 4**

Type 3: two digits, two divisors. 72 = 8 × 9. Fix the last digit with the 8-test, then the other with the 9-test.

Example 6. The number 6x31y is divisible by 72. Find x + y. (a) 6 (b) 8 (c) 10 (d) 12 *Solution.* For 8: 31y must be a multiple of 8; only 312 works, so y = 2. For 9: 6 + x + 3 + 1 + 2 = 12 + x, so x = 6. **Answer: (b) 8**

Type 4: multiples in a range. Multiples of n from a to b (both included) = ⌊b/n⌋ − ⌊(a − 1)/n⌋.

Example 7. How many multiples of 9 lie strictly between 200 and 500? *Solution.* ⌊499/9⌋ − ⌊200/9⌋ = 55 − 22. **Answer: 33**

Example 8. How many numbers from 1 to 300 are divisible by 4 but not by 6? *Solution.* Multiples of 4: 75. Divisible by both 4 and 6 means divisible by LCM 12: 25. 75 − 25 = 50. **Answer: 50**

Type 5: sum of multiples. Terms = (last − first)/n + 1; sum = terms × (first + last)/2.

Example 9. Find the sum of all three-digit multiples of 11. *Solution.* First 110, last 990; terms = 880/11 + 1 = 81. Sum = 81 × 1,100/2 = 81 × 550. **Answer: 44,550**

Type 6: nearest, smallest and largest multiples. Divide, then move to the multiple below or above.

Example 10. (i) Which number nearest to 4,000 is divisible by 23? (ii) What is the least number to add to 2,500 to make it divisible by 47? *Solution.* (i) 4,000 = 23 × 173 + 21. The multiple below is 3,979 (21 away); above is 4,002 (2 away). (ii) 2,500 = 47 × 53 + 9, so add 47 − 9 = 38. **Answer: (i) 4,002 (ii) 38**

- Smallest five-digit multiple of 37: 10,000 = 37 × 270 + 10, so take 37 × 271 = **10,027**. Largest four-digit multiple of 29: 9,999 = 29 × 344 + 23, so 9,999 − 23 = **9,976**.

Speed tips.

- For a missing digit, use the strictest test first (8, 11), then the digit-sum test.
- For "how many between", write the floor formula once; never list the numbers.
- "Strictly between" excludes both ends; "from … to" includes both.

Common traps

✗ "1 is a prime number." ✓ **1 is neither prime nor composite**; 2 is the smallest prime.

✗ "Co-prime numbers must both be prime." ✓ Co-prime means **HCF = 1** (8 and 9 are co-prime).

✗ "Divisible by 4 and 6, so divisible by 24." ✓ Only by **LCM 12**. Combine tests only for co-prime divisors.

✗ "Divisible by 6 and 9: count multiples of 54." ✓ Count multiples of the **LCM, 18**.

✗ "Multiples of 9 between 200 and 500 = 500/9 − 200/9 ≈ 33.3." ✓ Use whole parts: ⌊499/9⌋ − ⌊200/9⌋ = **33**.

✗ "Place value of 6 in 46,218 is 6." ✓ That is the face value; the place value is **6,000**.

Quick revision

- 1 is neither prime nor composite; 2 is the only even prime.
- 25 primes below 100; 15 below 50.
- Rational = terminating or recurring decimal.
- 4: last two digits; 8: last three; 16: last four.
- 3 and 9: digit sum. 11: alternate-digit difference 0 or 11k.
- Combine tests only for co-prime divisors.
- Multiples of n in [a, b] = ⌊b/n⌋ − ⌊(a − 1)/n⌋.
- "Divisible by both" means divisible by the LCM.
- abcabc = abc × 1,001 = abc × 7 × 11 × 13.
- Check primes only up to √n.

2 Remainders, unit digits and factors

Unit digits repeat in short cycles, remainders can be worked on small pieces, and the number of factors comes straight from the prime factorisation. (Factorials, factor counting and remainder theorems each have a fuller note in this unit: MAB-NS-09, 10 and 11.)

Core facts

- **Unit-digit cycles:**

Last digit	Cycle	Length
0, 1, 5, 6	same digit always	1
4	4, 6 (odd power 4, even power 6)	2
9	9, 1 (odd power 9, even power 1)	2
2	2, 4, 8, 6	4
3	3, 9, 7, 1	4
7	7, 9, 3, 1	4
8	8, 4, 2, 6	4

- **Rule:** divide the power by 4 and use the remainder; a remainder of **0 means the 4th term**.
- **Dividend = divisor × quotient + remainder**; the remainder is always less than the divisor.
- If N = p^a × q^b × r^c (p, q, r primes): **number of factors = (a + 1)(b + 1)(c + 1)**.
- **Trailing zeros** of a product = the smaller of the counts of 2s and 5s.
- $a^n + b^n$ is divisible by (a + b) when **n is odd**; $a^n - b^n$ is divisible by (a − b) always, and by (a + b) when n is even.

Example 1. Find the unit digit of 8^47. *Solution.* 47 ÷ 4 leaves 3; the 3rd term of 8, 4, 2, 6 is 2. **Answer: 2**

Example 2. Find the unit digit of 4^63. *Solution.* The power is odd, so the unit digit is 4. **Answer: 4**

Example 3. How many factors does 200 have? *Solution.* 200 = $2^3 \times 5^2$. Factors = 4 × 3. **Answer: 12**

Explained

Type 1: unit digit of a product or sum. Reduce each base to its last digit and each power modulo 4, then multiply or add the unit digits.

Example 4. Unit digit of 2^37 × 7^54 × 9^21. (a) 2 (b) 4 (c) 6 (d) 8 *Solution.* 2^37: 37 ÷ 4 leaves 1 → 2. 7^54: 54 ÷ 4 leaves 2 → 9. 9^21: odd power → 9. 2 × 9 × 9 = 162. **Answer: (a) 2**

Example 5. Unit digit of 3^50 + 7^50. *Solution.* 50 ÷ 4 leaves 2: 3^50 ends in 9, 7^50 ends in 9. 9 + 9 = 18. **Answer: 8**

Example 6. Unit digit of (1,237)^89 × (564)^22. *Solution.* Only last digits matter: 7^89 (89 ÷ 4 leaves 1) → 7; 4^22 (even) → 6. 7 × 6 = 42. **Answer: 2**

Type 2: remainder of a power. Find a small power that leaves 1 (or −1), then only the leftover power matters.

Example 7. Remainder when 3^41 is divided by 8. *Solution.* $3^2 = 9$ leaves 1. 3^41 = (3^2)^20 × 3, so the remainder is 3. **Answer: 3**

Example 8. Remainder when 5^63 is divided by 6. *Solution.* 5 leaves −1 on 6; (−1)^63 = −1, that is 6 − 1. **Answer: 5**

Type 3: remainder carried to a factor of the divisor.

Example 9. A number leaves 64 when divided by 255. What does it leave when divided by 15? *Solution.* 15 divides 255, so the answer is 64 ÷ 15's remainder. **Answer: 4**

- This works only when the new divisor is a **factor** of the old one.

Type 4: odd, even and total factors. Odd factors ignore the power of 2; even factors = total − odd.

Example 10. How many factors of 1,800 are even? (a) 9 (b) 18 (c) 27 (d) 36 *Solution.* 1,800 = $2^3 \times 3^2 \times 5^2$. Total = 4 × 3 × 3 = 36. Odd = 3 × 3 = 9. Even = 36 − 9. **Answer: (c) 27**

Speed tips.

- Write the four-term cycle once at the top of the rough sheet; most unit-digit questions then take one line.
- A factor count for a non-square is always **even**; an odd count means the number is a perfect square.

Common traps

✗ "8^48 ends in 8 because 48 ÷ 4 leaves 0, so take the first term." ✓ Remainder 0 means the **4th term**: 8^48 ends in **6**.

✗ "Number of factors of $2^3 \times 5^2$ = 3 × 2 = 6." ✓ Add 1 to each exponent: 4 × 3 = **12**.

✗ "N ÷ 255 leaves 64, so N ÷ 15 leaves 64." ✓ The remainder must be less than 15: **4**.

✗ "Even factors of 1,800 = 36 ÷ 2 = 18." ✓ Even = total − odd = 36 − 9 = **27**.

✗ "$a^n + b^n$ is always divisible by a + b." ✓ Only when **n is odd**.

Quick revision

- Cycles: 2, 3, 7, 8 → length 4; 4, 9 → length 2; 0, 1, 5, 6 → fixed.
- Power ÷ 4 remainder 0 means the 4th term of the cycle.
- Factors of p^a q^b r^c = (a+1)(b+1)(c+1).
- Odd factors: drop the power of 2. Even = total − odd.
- Only perfect squares have an odd number of factors.

- For remainders of powers, find a power leaving 1 or −1.
- Remainder carries over only to a factor of the divisor.
- Successive division: build the number from the last quotient outwards.
- $a^n + b^n \div (a + b)$ when n is odd.
- $1! + 2! + \ldots + n!$ $(n \geq 4)$ ends in 3.

3 HCF and LCM: methods and word problems

HCF is the largest number that divides all the given numbers, LCM is the smallest number they all divide, and most word problems only ask which of the two is needed and how the remainder is adjusted.

Core facts

- **HCF** (highest common factor, also GCD): the greatest number that divides each given number exactly.
- **LCM** (least common multiple): the smallest number exactly divisible by each given number.
- **Prime-factor method:** HCF = common primes with the **lowest** powers; LCM = all primes with the **highest** powers.
- **Division (Euclid) method:** divide the larger by the smaller, then the divisor by the remainder, until the remainder is 0. The last divisor is the HCF.
- **For two numbers only:** HCF × LCM = **product of the numbers**. HCF always divides the LCM.
- **Fractions** (in lowest terms): HCF = HCF of numerators ÷ LCM of denominators; LCM = LCM of numerators ÷ HCF of denominators.
- **Decimals:** give all numbers the same number of decimal places, work with the whole numbers, then put the decimal places back.
- **Key words:** "greatest", "largest size", "maximum number of equal groups" → **HCF**. "Least", "together again", "first time all" → **LCM**.

Example 1. Find the HCF and LCM of 72 and 120. *Solution.* $72 = 2^3 \times 3^2$; $120 = 2^3 \times 3 \times 5$. HCF = $2^3 \times 3$ = 24; LCM = $2^3 \times 3^2 \times 5$ = 360. **Answer: 24 and 360**

Example 2. The product of two numbers is 4,320 and their LCM is 360. Find the HCF. *Solution.* HCF = 4,320 ÷ 360. **Answer: 12**

Example 3. Find the HCF of 493 and 899 by division. *Solution.* 899 = 1 × 493 + 406; 493 = 1 × 406 + 87; 406 = 4 × 87 + 58; 87 = 1 × 58 + 29; 58 = 2 × 29. **Answer: 29**

Explained

Type 1: ratio with HCF or LCM. Write the numbers as ak and bk with a, b co-prime; then HCF = k and LCM = abk.

Example 4. Two numbers are in the ratio 4 : 5 and their LCM is 420. Find their sum. (a) 168 (b) 189 (c) 210 (d) 231 *Solution.* LCM = 20k = 420, so k = 21. Numbers 84 and 105. **Answer: (b) 189**

Type 2: least number leaving the same remainder. Answer = LCM × k + remainder (k = 1 for the least).

Example 5. Least number that leaves 5 when divided by 8, 12, 18 and 30. *Solution.* LCM(8, 12, 18, 30) = $2^3 \times 3^2 \times 5$ = 360. 360 + 5. **Answer: 365**

Type 3: greatest number leaving given remainders. Subtract each remainder, then take the HCF.

Example 6. Greatest number that divides 1,342 and 2,058 leaving remainders 7 and 11. *Solution.* 1,342 − 7 = 1,335 = 89 × 15; 2,058 − 11 = 2,047 = 89 × 23. HCF = 89, which is larger than both remainders. **Answer: 89**

Type 4: greatest number leaving the same (unknown) remainder. Take the HCF of the **differences**.

Example 7. Greatest number that divides 1,277, 3,167 and 6,002 leaving the same remainder each time. *Solution.* Differences: 1,890, 2,835 and 4,725. HCF = 945. (Each number leaves 332.) **Answer: 945**

Type 5: tiles, rods and equal groups (HCF).

Example 8. A floor 6.24 m by 4.68 m is to be covered with the largest possible square tiles, no cutting. How many tiles are needed? *Solution.* In cm: HCF(624, 468) = 156, so tiles are 1.56 m square. Tiles = (624 ÷ 156) × (468 ÷ 156) = 4 × 3. **Answer: 12**

Type 6: bells, lights and runners (LCM).

Example 9. Three traffic lights change every 40, 56 and 84 seconds. They change together at 7:45:00 a.m. When do they next change together? *Solution.* LCM(40, 56, 84) = $2^3 \times 3 \times 5 \times 7$ = 840 seconds = 14 min. **Answer: 7:59:00 a.m.**

Example 10. Four bells ring every 9, 12, 15 and 18 minutes and ring together at noon. How many times do they ring together in the 6 hours from noon, counting noon? *Solution.* LCM = 180 minutes. 360 ÷ 180 = 2 intervals, so 2 + 1 times. **Answer: 3**

Speed tips.

- Decide HCF or LCM from the key word before calculating.
- "Greatest … remainders": subtract the remainders first. "Least … remainder": add it after.
- In "least number" MCQs, test the smallest option against the last condition first.

Common traps

✗ "Bells ring together 2 times in 6 hours." ✓ Count the start too: **3** when the question says "counting noon".

✗ "HCF × LCM = product of three numbers." ✓ The rule holds only for **two** numbers.

✗ "Greatest number leaving remainders 7 and 11: take HCF of 1,342 and 2,058." ✓ Subtract the remainders first: HCF of **1,335 and 2,047**.

✗ "Same remainder each time: take the HCF of the numbers." ✓ Take the HCF of their **differences**.

✗ "LCM of fractions = LCM of numerators ÷ LCM of denominators." ✓ Divide by the **HCF** of denominators.

✗ "Two numbers can have HCF 12 and LCM 90." ✓ The LCM must be a **multiple of the HCF**; no such pair exists.

Quick revision

- HCF: lowest powers of common primes. LCM: highest powers of all primes.
- Two numbers: HCF × LCM = product; HCF divides LCM.
- Fractions: HCF = HCF(num)/LCM(den); LCM = LCM(num)/HCF(den).
- Same remainder r: answer = LCM × k + r.
- Constant shortfall d: LCM − d.
- Greatest divisor with given remainders: HCF of (number − remainder).
- Greatest divisor with the same unknown remainder: HCF of differences.
- Tiles and equal groups: HCF. Bells and lights: LCM, add 1 for the start.
- Pairs with HCF h, LCM l: co-prime pairs with product l/h.

4 Simplification and approximation: BODMAS, surds, indices and roots

Simplification questions test order of operations, a handful of identities and the habit of spotting a pattern; most collapse to a small whole number once the pattern is seen. (Decimals, roots, and surds and indices have fuller notes: MAB-NS-06, 07 and 08.)

Core facts

- **BODMAS:** Brackets → Of → Division and Multiplication (left to right) → Addition and Subtraction (left to right). "Of" means multiply.
- Brackets open from the inside: ( ) first, then { }, then [ ]. A bar (vinculum) over terms acts as a bracket.
- **Identities:**
  - $(a + b)^2 = a^2 + 2ab + b^2$; $(a - b)^2 = a^2 - 2ab + b^2$; $a^2 - b^2 = (a + b)(a - b)$
  - $(a + b)^2 - (a - b)^2 = 4ab$; $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$
  - $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$; $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
  - **If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.**
- **Laws of indices:** $a^m \times a^n = a^{m+n}$; $(a^m)^n = a^{mn}$; $\mathbf{a^0 = 1}$; $a^{-n} = 1/a^n$.
- **Rationalising:** $1/(\sqrt{a} - \sqrt{b}) = (\sqrt{a} + \sqrt{b})/(a - b)$.
- **Series sums:** $1 + 2 + \ldots + n = n(n+1)/2$; $1^2 + \ldots + n^2 = n(n+1)(2n+1)/6$; $1^3 + \ldots + n^3 = [n(n+1)/2]^2$.

Example 1. Simplify 48 ÷ 6 × 2 + 3 of 4 − 5. *Solution.* "Of" first: 3 of 4 = 12. Then left to right: 48 ÷ 6 = 8, × 2 = 16. 16 + 12 − 5. **Answer: 23**

Example 2. Simplify 20 − [8 − {6 − (4 − 1)}]. *Solution.* (4 − 1) = 3; {6 − 3} = 3; [8 − 3] = 5; 20 − 5. **Answer: 15**

Example 3. Find 103 × 97 without long multiplication. *Solution.* (100 + 3)(100 − 3) = 10,000 − 9. **Answer: 9,991**

Explained

Type 1: identity in disguise. Large numbers placed symmetrically almost always fit an identity.

Example 4. Value of $[(467 + 133)^2 - (467 - 133)^2] \div (467 \times 133)$. (a) 2 (b) 4 (c) 600 (d) 334 *Solution.* $(a + b)^2 - (a - b)^2 = 4ab$, so the quotient is 4. **Answer: (b) 4**

Example 5. Value of $(0.7^3 + 0.3^3) \div (0.7^2 - 0.7 \times 0.3 + 0.3^2)$. *Solution.* $(a^3 + b^3)/(a^2 - ab + b^2) = a + b = 0.7 + 0.3$. **Answer: 1**

Example 6. Value of $17^3 - 9^3 - 8^3$. *Solution.* Take a = 17, b = −9, c = −8; a + b + c = 0. So the value = 3abc = 3 × 17 × (−9) × (−8). **Answer: 3,672**

Type 2: find the missing number. Simplify each side separately, then solve.

Example 7. 25% of 480 + ? = 3/5 of 350. *Solution.* 120 + ? = 210. **Answer: 90**

Type 3: approximation. Round each number to an easy value; options are usually far apart.

Example 8. 39.97% of 601.8 + 14.02 × 9.98 ≈ ? (a) 320 (b) 350 (c) 380 (d) 420 *Solution.* 40% of 600 = 240; 14 × 10 = 140; 240 + 140 = 380. (Exact value 380.46.) **Answer: (c) 380**

Type 4: telescoping sums and products. Split each term so that neighbours cancel.

Example 9. 1/(2×3) + 1/(3×4) + … + 1/(29×30). *Solution.* 1/(n(n+1)) = 1/n − 1/(n+1). Sum = 1/2 − 1/30 = 14/30. **Answer: 7/15**

Example 10. (1 − 1/3)(1 − 1/4)(1 − 1/5) … (1 − 1/40). *Solution.* = (2/3)(3/4)(4/5) … (39/40). Everything cancels except 2/40. **Answer: 1/20**

Speed tips.

- Before calculating, look for a^2, b^2, 2ab or a^3, b^3 patterns; the numbers are chosen to fit one.
- In approximation questions, check how far apart the options are before deciding how roughly to round.
- Division and multiplication go left to right; never do × before ÷ just because "M" is in the word.

Common traps

✗ "8 ÷ 2 × 4 = 1 because multiplication comes first." ✓ Division and multiplication go **left to right**: (8 ÷ 2) × 4 = **16**.

✗ "$x^2 + 1/x^2 = (x + 1/x)^2$." ✓ Subtract 2: $(x + 1/x)^2 - \mathbf{2}$.

✗ "$x^3 + 1/x^3 = (x + 1/x)^3$." ✓ Subtract 3(x + 1/x).

✗ "√(a + b) = √a + √b." ✓ That holds only for products: √(ab) = √a × √b.

✗ "1/(1×4) + … telescopes to 1 − 1/31." ✓ Divide by the gap 3: **(1/3)(1 − 1/31)**.

✗ "$a^0 = 0$." ✓ $\mathbf{a^0 = 1}$ for any non-zero a.

Quick revision

- BODMAS; "of" before ÷; ÷ and × left to right.
- $(a + b)^2 - (a - b)^2 = 4ab$.
- $a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$.
- $x^2 + 1/x^2 = (x + 1/x)^2 - 2$; $x^3 + 1/x^3 = (x + 1/x)^3 - 3(x + 1/x)$.
- $\Sigma n = n(n+1)/2$; $\Sigma n^2 = n(n+1)(2n+1)/6$; $\Sigma n^3 = [n(n+1)/2]^2$.
- $1/(n(n+k)) = (1/k)(1/n - 1/(n+k))$.
- Products like (1 − 1/n) telescope to first numerator ÷ last denominator.
- Continued fractions: simplify from the bottom up.
- Compare fractions close to 1 by their gap from 1.

- Approximate only as roughly as the option gaps allow.

5 Number puzzles: digits, sums and differences

Write a number in place-value form, turn the words into one or two equations, and most digit puzzles solve in two lines. (Counting digits and page numbers is in MAB-NS-14; general word problems are in MAB-NS-12.)

Core facts

- A **two-digit number** with tens digit a and units digit b is **10a + b**; reversed, it is 10b + a.
  - Sum with its reverse = **11(a + b)**.
  - Difference from its reverse = **9(a − b)**.
- A **three-digit number** abc = 100a + 10b + c. abc − cba = **99(a − c)**.
- **Consecutive numbers:** x, x + 1, x + 2. **Consecutive even or odd:** x, x + 2, x + 4.
- The middle of an odd count of consecutive numbers = **sum ÷ count**.
- **Digits in AP:** for three digits, the middle digit = digit sum ÷ 3.
- In cryptarithms, different letters are **different digits** and a leading letter is never 0.

Example 1. A two-digit number has digit sum 9. Reversing the digits increases it by 27. Find the number. *Solution.* 9(b − a) = 27, so b − a = 3; a + b = 9. a = 3, b = 6. **Answer: 36**

Example 2. A two-digit number exceeds its reverse by 63, and its digits add to 11. Find it. *Solution.* 9(a − b) = 63, so a − b = 7; a + b = 11. a = 9, b = 2. **Answer: 92**

Example 3. Three consecutive odd numbers add to 87. Find them. *Solution.* Middle = 87 ÷ 3 = 29. **Answer: 27, 29, 31**

Explained

Type 1: number as a multiple of its digit sum. Expand and simplify to a relation between digits.

Example 4. A two-digit number is 7 times the sum of its digits, and it exceeds its reverse by 36. Find it. (a) 42 (b) 63 (c) 84 (d) 73 *Solution.* 10a + b = 7a + 7b gives a = 2b. 9(a − b) = 36 gives a − b = 4. So b = 4, a = 8. **Answer: (c) 84**

Type 2: digit product with reversal.

Example 5. The product of the digits of a two-digit number is 18. Subtracting 27 reverses the digits. Find the number. *Solution.* 9(a − b) = 27, so a = b + 3. (b + 3)b = 18 gives b = 3, a = 6. **Answer: 63**

Type 3: three-digit numbers.

Example 6. The digits of a three-digit number are in AP and add to 12. Subtracting 396 reverses the number. Find it. *Solution.* Middle digit = 12 ÷ 3 = 4. 99(a − c) = 396, so a − c = 4; a + c = 8. a = 6, c = 2. **Answer: 642**

Type 4: sum of all numbers formed by permuting digits. With n distinct digits, each digit sits in each place (n − 1)! times. Sum = (digit sum) × (n − 1)! × 11…1 (n ones).

Example 7. Find the sum of all three-digit numbers formed with 2, 4 and 7, each used once. *Solution.* (2 + 4 + 7) × 2! × 111 = 13 × 2 × 111. **Answer: 2,886**

Type 5: operator puzzles. Find the rule from two cases, confirm it on the third, then apply.

Example 8. If 2 * 5 = 14, 3 * 4 = 21 and 5 * 2 = 35, find 6 * 3. (a) 45 (b) 54 (c) 63 (d) 36 *Solution.* Rule $a^2 + ab$: 4 + 10 = 14, 9 + 12 = 21, 25 + 10 = 35. So 36 + 18 = 54. **Answer: (b) 54**

Example 9. For whole numbers, a ◊ b = a − b if a > b, a + b if a < b, and ab if a = b. Find (7 ◊ 4) ◊ (3 ◊ 3). *Solution.* 7 ◊ 4 = 3; 3 ◊ 3 = 9; 3 ◊ 9 = 3 + 9 (since 3 < 9). **Answer: 12**

Speed tips.

- Use 11(a + b) and 9(a − b) directly instead of building full equations.
- In MCQs, test the options against the easiest condition first (the reversal difference), then the second.
- An operator rule must fit **every** example given, not just two.

Common traps

✗ "abc − cba = 9(a − c)." ✓ For three digits it is **99(a − c)**.

✗ "Three consecutive odd numbers: x, x + 1, x + 2." ✓ They step by 2: **x, x + 2, x + 4**.

✗ "Sum of permutations of 2, 4, 7 = 13 × 111." ✓ Each digit appears in each place **2! = 2** times: 13 × 2 × 111 = **2,886**.

✗ "Three-digit numbers with digit sum 6 = C(8, 2) = 28." ✓ The hundreds digit cannot be 0: **C(7, 2) = 21**.

✗ "The rule fits the first two examples, so it is right." ✓ Check it on **every** example.

Quick revision

- Two-digit: 10a + b; reverse 10b + a.
- Sum with reverse 11(a + b); difference 9(a − b).
- Three-digit difference with reverse 99(a − c).
- Middle of consecutive numbers = sum ÷ count.
- Digits in AP: middle digit = sum ÷ 3.
- Permutation sum = (digit sum) × (n − 1)! × 11…1.
- Digit-sum counting: shift the leading digit by 1, then C(s + 2, 2).
- Cryptarithm: start from the units digit and the size of the result.
- Operator rule must fit every example.

6 Decimal fractions and recurring decimals: conversion and comparison

Every fraction is a terminating or a recurring decimal; know when each happens, convert both ways by rule, and handle decimal points by counting places rather than guessing.

Core facts

- **Decimal fraction:** a fraction whose denominator is a power of 10 (0.07 = 7/100).
- **Decimal to fraction:** write the digits over 1 followed by as many zeros as there are decimal places, then reduce: 0.875 = 875/1,000 = 7/8.
- **Terminating test:** a fraction **in lowest terms** terminates only if its denominator has no prime factor other than **2 and 5**. The number of decimal places = the larger of the powers of 2 and 5.
- **Notation:** in this note, digits in brackets recur: 0.(36) = 0.363636...; 0.2(45) = 0.2454545...
- **Pure recurring** (all digits after the point recur): numerator = the recurring block; denominator = as many **9s** as digits in the block. 0.(7) = 7/9; 0.(36) = 36/99.
- **Mixed recurring:** numerator = (all digits after the point) − (non-recurring digits); denominator = as many 9s as recurring digits, followed by as many 0s as non-recurring digits. 0.2(45) = (245 − 2)/990.
- **Multiplying decimals:** multiply as whole numbers; decimal places in the answer = **sum** of decimal places.
- **Dividing decimals:** shift the point in both numbers until the divisor is a whole number.
- Adding zeros at the right end of a decimal does not change it: 0.5 = 0.50 = 0.500.

Example 1. Write 0.875 as a fraction in lowest terms. *Solution.* 875/1,000; divide by 125. **Answer: 7/8**

Example 2. Write 0.(36) as a fraction. *Solution.* 36/99 = 4/11. **Answer: 4/11**

Example 3. Find 0.18 × 0.05. *Solution.* 18 × 5 = 90; places 2 + 2 = 4, so 0.0090. **Answer: 0.009**

Explained

Type 1: mixed recurring to a fraction.

Example 4. Write 0.2(45) as a fraction in lowest terms. (a) 27/110 (b) 243/999 (c) 245/990 (d) 27/99 *Solution.* (245 − 2)/990 = 243/990; divide by 9: 27/110. **Answer: (a) 27/110**

Example 5. Write 3.1(6) as a fraction. *Solution.* Whole part 3 stays. 0.1(6) = (16 − 1)/90 = 15/90 = 1/6. So 3 + 1/6. **Answer: 19/6**

Type 2: will it terminate, and after how many places? Reduce the fraction first.

Example 6. Which of these does NOT give a terminating decimal? (a) 17/80 (b) 21/56 (c) 9/75 (d) 13/60 *Solution.* 17/80: 80 = 2^4 × 5, terminates. 21/56 = 3/8, terminates. 9/75 = 3/25, terminates. 13/60: 60 = 2^2 × 3 × 5 has a 3. **Answer: (d) 13/60**

Example 7. After how many decimal places does $23/(2^3 \times 5^4)$ terminate? *Solution.* The larger power is 4, so 4 places (it equals 0.0046). **Answer: 4**

Type 3: dividing decimals.

Example 8. Find 0.0084 ÷ 0.12. *Solution.* Multiply both by 100: 0.84 ÷ 12 = 0.07. **Answer: 0.07**

Type 4: arranging fractions. Convert each to a decimal to 3 places, or compare pairs by cross-multiplying.

Example 9. Arrange 4/7, 5/9, 7/12 and 9/16 in ascending order. *Solution.* 4/7 ≈ 0.571, 5/9 ≈ 0.556, 7/12 ≈ 0.583, 9/16 = 0.5625. **Answer: 5/9 < 9/16 < 4/7 < 7/12**

Type 5: sums of recurring decimals. Convert to fractions, add, convert back if needed.

Example 10. Find 0.(4) + 0.(5) + 0.(6). *Solution.* (4 + 5 + 6)/9 = 15/9 = 5/3 = 1.(6). **Answer: 5/3**

Example 11. Simplify (0.6 × 0.6 − 0.4 × 0.4) ÷ (0.6 − 0.4). *Solution.* $(a^2 - b^2)/(a - b) = a + b = 1$. **Answer: 1**

Speed tips.

- Learn the common decimals: 1/8 = 0.125, 3/8 = 0.375, 5/8 = 0.625, 7/8 = 0.875, 1/16 = 0.0625, 1/7 = 0.(142857), 1/9 = 0.(1), 1/11 = 0.(09), 1/12 = 0.08(3).
- Counting decimal places is safer than "moving the point" by eye; write the count down.

Common traps

✗ "0.2(45) = 245/999." ✓ Only "45" recurs: (245 − 2)/**990** = 27/110.

✗ "21/56 does not terminate because 56 has a 7." ✓ Reduce first: 21/56 = **3/8**, which terminates.

✗ "0.18 × 0.05 = 0.09." ✓ Places add: 2 + 2 = 4, so **0.0090 = 0.009**.

✗ "0.(9) is slightly less than 1." ✓ **0.(9) = 1** exactly.

✗ "0.0084 ÷ 0.12 = 0.7." ✓ Shift both points two places: 0.84 ÷ 12 = **0.07**.

Quick revision

- Terminates iff the reduced denominator has only 2s and 5s.
- Decimal places = larger of the powers of 2 and 5 in the denominator.
- Pure recurring: block over as many 9s.
- Mixed recurring: (all digits − non-recurring digits) over 9s then 0s.
- Product of decimals: add the decimal places.
- Division: make the divisor a whole number first.
- 0.(ab) + 0.(ba) = (a + b)/9.
- 1/7 = 0.(142857); nth digit from n mod 6.
- 0.(9) = 1.

- 1/8 = 0.125, 1/16 = 0.0625, 1/12 = 0.08(3).

7 Squares, square roots, cubes and cube roots: fast methods and estimation

Factorise to find exact roots, use the last digit and the leading group to read roots of large perfect squares and cubes in seconds, and estimate the rest with a ± b/(2a).

Core facts

- Learn **squares to 30** and **cubes to 15**. Useful ones: $13^2 = 169$, $17^2 = 289$, $19^2 = 361$, $23^2 = 529$, $27^2 = 729$; $11^3 = 1{,}331$, $12^3 = 1{,}728$, $13^3 = 2{,}197$, $14^3 = 2{,}744$, $15^3 = 3{,}375$.
- **Square root by factorisation:** pair the prime factors; take one from each pair. **Cube root:** group in threes.
- A **perfect square** has every prime exponent even; a **perfect cube** has every exponent a multiple of 3.
- **Last digit of squares:** only 0, 1, 4, 5, 6, 9. A square never ends in 2, 3, 7 or 8, or in an odd number of zeros.

Square ends in	Root ends in
1	1 or 9
4	2 or 8
9	3 or 7
6	4 or 6
5	5

- **Cubes** keep the last digit for 0, 1, 4, 5, 6, 9 and swap **2 ↔ 8** and **3 ↔ 7**.
- **Digits:** a square of an n-digit number has 2n − 1 or 2n digits. A root of a 2k-digit or (2k − 1)-digit square has k digits.
- **Roots of decimals:** the number of decimal places must be even (square) or a multiple of 3 (cube); the root has half (or a third) as many: $\sqrt{0.0169} = 0.13$.

Example 1. Find $\sqrt{5{,}184}$. *Solution.* $5{,}184 = 2^6 \times 3^4$. Root $= 2^3 \times 3^2 = 8 \times 9$. **Answer: 72**

Example 2. Find $\sqrt[3]{2{,}744}$. *Solution.* $2{,}744 = 2^3 \times 7^3$. Root $= 2 \times 7$. **Answer: 14**

Example 3. Least number by which 1,176 must be multiplied to make a perfect square. *Solution.* $1{,}176 = 2^3 \times 3 \times 7^2$. Make every exponent even: multiply by 2×3. **Answer: 6**

Explained

Type 1: square root of a large perfect square (no long division). Drop the last two digits; find the largest square at or below what is left; that gives the leading part. The last digit gives two choices; pick by comparing with n(n + 1).

Example 4. Find √119,716. *Solution.* Last digit 6, so the root ends in 4 or 6. Drop "16": 1,197. $34^2 = 1{,}156 \le 1{,}197 < 35^2 = 1{,}225$, so the root is 34_. 34 × 35 = 1,190 < 1,197, so take the larger ending, 6. **Answer: 346**

Type 2: cube root of a perfect cube. Last digit by the swap rule; drop the last three digits and find the largest cube at or below the rest.

Example 5. Find $\sqrt[3]{438{,}976}$. *Solution.* Ends in 6, so the root ends in 6. Drop "976": 438. $7^3 = 343 \le 438 < 8^3 = 512$. **Answer: 76**

Type 3: roots of decimals.

Example 6. Find √0.000441. *Solution.* Six decimal places, so the root has three. √441 = 21, giving 0.021. **Answer: 0.021**

Type 4: estimation. $\sqrt{a^2 + b} \approx a + b/(2a)$ for small b.

Example 7. Estimate √83 to two decimal places. (a) 9.06 (b) 9.11 (c) 9.16 (d) 9.21 *Solution.* 83 = 81 + 2: 9 + 2/18 ≈ 9.11. (True value 9.110.) **Answer: (b) 9.11**

Example 8. Given √5 = 2.236, find 1/√5. *Solution.* 1/√5 = √5/5 = 2.236 ÷ 5. **Answer: 0.4472**

- Always rationalise before dividing; dividing 1 by 2.236 is slow.

Type 5: least number to add or subtract.

Example 9. (i) Least number to subtract from 6,000 to get a perfect square. (ii) Least number to add. *Solution.* $77^2 = 5{,}929$ and $78^2 = 6{,}084$. (i) 6,000 − 5,929 = 71. (ii) 6,084 − 6,000 = 84. **Answer: (i) 71 (ii) 84**

Type 6: spot the perfect square. Use the last digit, then the tens digit, then the digital root.

Example 10. Only one is a perfect square: (a) 6,246 (b) 6,241 (c) 6,248 (d) 6,251 *Solution.* (c) ends in 8: rejected. A square ending in 6 has an **odd** tens digit, so (a) is out. A square ending in 1 has an **even** tens digit, so (d) is out. $79^2 = 6{,}241$. **Answer: (b) 6,241**

Speed tips.

- Squares ending in 1, 4, 9 or 5 have an even tens digit (25 for 5); squares ending in 6 have an odd tens digit.
- Digital root (repeated digit sum) of a square is 1, 4, 7 or 9.

Common traps

✗ "√0.0169 = 0.013." ✓ Four decimal places give two in the root: **0.13**.

✗ "6,246 ends in 6, so it may be a square." ✓ A square ending in 6 has an **odd tens digit**; 6,246 is not a square.

✗ "The cube root of 438,976 ends in 4." ✓ Cubes keep 6 → root ends in **6** (76). Only 2 ↔ 8 and 3 ↔ 7 swap.

✗ "Squares or cubes up to 1,000 = 31 + 10 = 41." ✓ Subtract the 3 sixth powers counted twice: **38**.

✗ "To make 1,176 a square, multiply by 3." ✓ The 2 also has an odd exponent (2^3): multiply by **6**.

Quick revision

- Squares end only in 0, 1, 4, 5, 6, 9.
- Root ends: 1 → 1/9, 4 → 2/8, 9 → 3/7, 6 → 4/6, 5 → 5.
- Cube roots: 2 ↔ 8 and 3 ↔ 7 swap; others stay.
- Large square: drop two digits, find the leading square, choose with $n(n + 1)$.
- Large cube: drop three digits, find the leading cube.
- $\sqrt{(a^2 + b)} \approx a + b/(2a)$.
- Square: all exponents even. Cube: all multiples of 3.
- Tens digit odd only for squares ending in 6.
- Digital root of a square: 1, 4, 7, 9.
- $\sqrt{(n + \sqrt{(n + \ldots)})} = k$ when $n = k(k - 1)$.

8 Surds and indices: laws, rationalising and comparing surds

Bring every term to a common base to solve index questions, rationalise with the conjugate, and compare surds by raising them to a common power.

Core facts

- **Laws of indices** ($a, b > 0$):
  - $a^m \times a^n = a^{m+n}$; $a^m \div a^n = a^{m-n}$; $(a^m)^n = a^{mn}$
  - $(ab)^n = a^n b^n$; $(a/b)^n = a^n/b^n$
  - $\mathbf{a^0 = 1}$; $a^{-n} = 1/a^n$; $a^{(m/n)} = (\sqrt[n]{a})^m = \sqrt[n]{(a^m)}$
- If $a^x = a^y$ ($a > 0$, $a \neq 1$), then $\mathbf{x = y}$.
- **Surd:** an irrational root of a rational number, such as $\sqrt{3}$ or $\sqrt[3]{5}$. $\sqrt{4}$ is not a surd.
- **Order of a surd:** the root index. $\sqrt[3]{5}$ is a surd of order 3.
- **Like surds** have the same irrational part after simplifying: $\sqrt{50} = 5\sqrt{2}$ and $\sqrt{18} = 3\sqrt{2}$ are like surds; only like surds can be added.
- **Rationalising factor:** for $\sqrt{a}$ it is $\sqrt{a}$; for $(\sqrt{a} + \sqrt{b})$ it is the **conjugate** $(\sqrt{a} - \sqrt{b})$, since $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$.
- **Comparing surds of different orders:** raise each to the LCM of the orders.

Example 1. Find $27^{(2/3)}$. *Solution.* $(\sqrt[3]{27})^2 = 3^2$. **Answer: 9**

Example 2. If $2^x = 1/32$, find x. *Solution.* $1/32 = 2^{-5}$. **Answer: −5**

Example 3. Simplify $\sqrt{50} + \sqrt{18} - \sqrt{8}$. *Solution.* $5\sqrt{2} + 3\sqrt{2} - 2\sqrt{2}$. **Answer: $6\sqrt{2}$**

Explained

Type 1: exponential equations. Write both sides with the same base, then equate powers.

Example 4. If $9^x = 27^{(x-1)}$, find x. (a) 1 (b) 2 (c) 3 (d) 4 *Solution.* $3^{(2x)} = 3^{(3x-3)}$, so $2x = 3x - 3$. **Answer: (c) 3**

Example 5. If $5^{(x+3)} = 25^{(3x-4)}$, find x. *Solution.* $5^{(x+3)} = 5^{(6x-8)}$, so $x + 3 = 6x - 8$, $5x = 11$. **Answer: 11/5**

Type 2: simplifying powers.

Example 6. Simplify $16^{(0.75)} \times 64^{(-1/3)}$. *Solution.* $16^{(3/4)} = 2^3 = 8$; $64^{(-1/3)} = 1/4$. $8 \times 1/4$. **Answer: 2**

Example 7. Simplify $(x^a/x^b)^{(a+b)} \times (x^b/x^c)^{(b+c)} \times (x^c/x^a)^{(c+a)}$. *Solution.* Powers: $(a - b)(a + b) + (b - c)(b + c) + (c - a)(c + a) = (a^2 - b^2) + (b^2 - c^2) + (c^2 - a^2) = 0$. $x^0 = 1$. **Answer: 1**

Type 3: rationalising.

Example 8. Rationalise $6/(\sqrt{5} + \sqrt{2})$. *Solution.* Multiply by $(\sqrt{5} - \sqrt{2})$: $6(\sqrt{5} - \sqrt{2})/(5 - 2) = 2(\sqrt{5} - \sqrt{2})$. **Answer: $2\sqrt{5} - 2\sqrt{2}$**

Type 4: square root of a surd expression. Write $a \pm 2\sqrt{b}$ as $(\sqrt{x} \pm \sqrt{y})^2$ with $x + y = a$ and $xy = b$.

Example 9. Find $\sqrt{(11 - 2\sqrt{30})}$. *Solution.* Need $x + y = 11$, $xy = 30$: $x = 6$, $y = 5$. Root $= \sqrt{6} - \sqrt{5}$ (positive, since $\sqrt{6} > \sqrt{5}$). **Answer: $\sqrt{6} - \sqrt{5}$**

Type 5: comparing surds of different orders.

Example 10. Which is largest: $\sqrt{3}$, $\sqrt[3]{4}$, $\sqrt[4]{6}$? *Solution.* LCM of 2, 3, 4 is 12. Raise each to the 12th power: $3^6 = 729$, $4^4 = 256$, $6^3 = 216$. **Answer: $\sqrt{3}$**

Speed tips.

- Convert decimals in indices to fractions at once: 0.75 = 3/4, 0.2 = 1/5, 1.5 = 3/2.
- In Type 4, the number under the inner root is the product xy, and the whole number is the sum x + y.

Common traps

✗ "$a^0 = 0$." ✓ **$a^0 = 1$** for any non-zero a.

✗ "$a^m \times b^m = (ab)^{2m}$." ✓ It is **$(ab)^m$**; powers add only for the same base.

✗ "$\sqrt{5} + \sqrt{3} = \sqrt{8}$." ✓ Only like surds add; $\sqrt{5} + \sqrt{3}$ cannot be simplified.

✗ "$\sqrt{(11 - 2\sqrt{30})} = \sqrt{5} - \sqrt{6}$." ✓ A square root is positive: **$\sqrt{6} - \sqrt{5}$**.

✗ "$\sqrt[4]{6} > \sqrt{3}$ because $6 > 3$." ✓ Compare at a common power: $6^3 = 216 < 3^6 = 729$, so **$\sqrt{3}$ is larger**.

✗ "$(2^3)^2 = 2^9$." ✓ $(a^m)^n = a^{mn} = \mathbf{2^6}$; $2^{(3^2)} = 2^9$ is a different expression.

Quick revision

- $a^m a^n = a^{m+n}$; $(a^m)^n = a^{mn}$; $a^{-n} = 1/a^n$; $a^{(m/n)} = (\sqrt[n]{a})^m$.
- Same base, equal powers: equate the exponents.
- Like surds only can be added or subtracted.
- Conjugate of $\sqrt{a} + \sqrt{b}$ is $\sqrt{a} - \sqrt{b}$; product $a - b$.
- $\sqrt{(a \pm 2\sqrt{b})} = \sqrt{x} \pm \sqrt{y}$ with $x + y = a$, $xy = b$.
- Compare surds at the LCM of their orders.
- Compare $\sqrt{a} + \sqrt{b}$ sums by squaring.
- $2^a = 3^b = 6^c \Rightarrow 1/a + 1/b = 1/c$.
- 0.75 = 3/4; 0.25 = 1/4; 1.5 = 3/2 in indices.

9 Factorials: trailing zeros and the highest power of a prime in n!

Count how many times a prime goes into n! by repeated division ($n/p + n/p^2 + \ldots$), and every question on zeros, powers and divisibility of factorials becomes a short sum.

Core facts

- **n! = 1 × 2 × 3 × … × n. 0! = 1** and 1! = 1.
- Values: 4! = 24, 5! = 120, 6! = 720, 7! = 5,040, 8! = 40,320, 10! = 36,28,800.
- **Highest power of a prime p in n!** (Legendre): $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \ldots$ (stop when $p^k > n$).
  - Short form: divide n by p, then divide the quotient by p again, and add all the quotients.
- **Trailing zeros of n!** = highest power of 5 in n! (2s are always more plentiful than 5s in n!).
- **Composite divisor:** split it into prime powers, find each limit, and take the **smallest**.
- From **5!** onwards every factorial ends in 0; from **10!** onwards in 00.

Example 1. How many zeros are at the end of 70!? *Solution.* 70/5 = 14; 14/5 = 2 (whole part). 14 + 2. **Answer: 16**

Example 2. How many zeros are at the end of 200!? *Solution.* 200/5 = 40; 40/5 = 8; 8/5 = 1. 40 + 8 + 1. **Answer: 49**

Example 3. Find 10!/7!. *Solution.* 10 × 9 × 8. **Answer: 720**

Explained

Type 1: highest power of a prime.

Example 4. Highest power of 3 that divides 80!. (a) 3^{26} (b) 3^{34} (c) 3^{36} (d) 3^{40} *Solution.* 80/3 = 26; 26/3 = 8; 8/3 = 2. 26 + 8 + 2 = 36. **Answer: (c) 3^{36}**

Example 5. Highest power of 7 in 100!. *Solution.* 100/7 = 14; 14/7 = 2. **Answer: 16**

Type 2: composite divisor. Count each prime, divide by its exponent in the divisor, take the smallest.

Example 6. Highest power of 12 that divides 30!. *Solution.* $12 = 2^2 \times 3$. Power of 2 in 30!: 15 + 7 + 3 + 1 = 26, giving $\lfloor 26/2 \rfloor = 13$ copies of 2^2. Power of 3: 10 + 3 + 1 = 14. Smaller is 13. **Answer: 13**

Example 7. Highest power of 15 in 75!. *Solution.* 15 = 3 × 5. Power of 5: 15 + 3 = 18. Power of 3: 25 + 8 + 2 = 35. The 5s are scarcer. **Answer: 18**

Type 3: zeros in a product of factorials. Add the zeros of each factorial (2s are plentiful in each).

Example 8. Zeros at the end of 25! × 30!. *Solution.* 25!: 5 + 1 = 6. 30!: 6 + 1 = 7. **Answer: 13**

Type 4: "exactly k zeros". The count jumps at each multiple of 25 (two 5s), 125 (three), and so on.

Example 9. For how many values of n does n! end in exactly 6 zeros? Is there an n! with exactly 5 zeros? *Solution.* 24! has 4 zeros; 25! has 5 + 1 = 6. The count stays 6 for 25!, 26!, 27!, 28!, 29! and becomes 7 at 30!. It jumps from 4 straight to 6, so no factorial ends in exactly 5 zeros. **Answer: 5 values (25 to 29); no**

Type 5: remainders and unit digits of factorial sums. Once k! is divisible by the divisor, all later terms drop out.

Example 10. Remainder when 1! + 2! + 3! + … + 100! is divided by 12. *Solution.* 4! = 24 and every later factorial is divisible by 12. So only 1 + 2 + 6 = 9 is left. **Answer: 9**

Speed tips.

- For trailing zeros, divide by 5 repeatedly in your head: 200 → 40 → 8 → 1.
- For a composite divisor, the **largest prime** is usually the scarce one, but check: in 30!, $12 = 2^2 \times 3$ is limited by 2^2, not by 3.

Common traps

✗ "Zeros in 200! = 200/5 = 40." ✓ Keep dividing: 40 + 8 + 1 = **49**.

✗ "Power of 3 in 80! = 80/3 = 26." ✓ Add 8 and 2 as well: **36**.

✗ "Highest power of 12 in 30! = power of 3 = 14." ✓ 2^2 needs two 2s per copy: $\lfloor 26/2 \rfloor$ = **13**, which is smaller.

✗ "Every whole number of zeros occurs for some n!." ✓ The count jumps at 25, 50, 75, 125, …; no n! ends in exactly **5** zeros.

✗ "In any product, zeros = number of 5s." ✓ That is true for n!, not for every product; count **both** 2s and 5s.

✗ "0! = 0." ✓ **0! = 1**.

Quick revision

- Power of prime p in n! = $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \ldots$
- Zeros of n! = power of 5 in n!.
- Composite divisor: limit for each prime power, take the smallest.
- 0! = 1; 5! = 120; 7! = 5,040; 10! = 36,28,800.
- 5! onwards ends in 0; 10! onwards ends in 00.
- 24! has 4 zeros, 25! has 6: no factorial has exactly 5.
- 250! has 62 zeros (249! has 59): the count jumps by at least 3 at each multiple of 125.
- Factorial sums: drop every term divisible by the divisor.
- Products of powers: count 2s and 5s separately.

10 Counting factors: number and sum of factors, perfect squares and co-primes

Write N as a product of prime powers once; the number, sum and product of its factors, and every "how many factors are …" question, then follow by counting choices of exponents.

Core facts

- Let **$N = p^a \times q^b \times r^c$** (p, q, r distinct primes).
- **Number of factors** = $(a + 1)(b + 1)(c + 1)$. (Each prime's exponent can be 0 to a.)
- **Sum of factors** = $(1 + p + \dots + p^a)(1 + q + \dots + q^b)(1 + r + \dots + r^c)$.
- **Product of factors** = N^(d/2), where d is the number of factors.
- **Odd factors:** ignore the power of 2. **Even factors** = total − odd.
- **Perfect squares** (and only they) have an **odd** number of factors.
- A number has **exactly 3 factors** only if it is the square of a prime.
- **Perfect number:** sum of its factors other than itself equals the number (6, 28, 496).
- **Co-primes up to N** (Euler's φ): $\varphi(N) = N \times (1 - 1/p)(1 - 1/q)(1 - 1/r)$.

Example 1. How many factors does 540 have? *Solution.* $540 = 2^2 \times 3^3 \times 5$. Factors = $3 \times 4 \times 2$. **Answer: 24**

Example 2. Find the sum of the factors of 72. *Solution.* $72 = 2^3 \times 3^2$. Sum = $(1 + 2 + 4 + 8)(1 + 3 + 9) = 15 \times 13$. **Answer: 195**

Example 3. Does 196 have an odd or even number of factors? *Solution.* $196 = 14^2 = 2^2 \times 7^2$, a perfect square: $3 \times 3 = 9$ factors. **Answer: odd (9)**

Explained

Type 1: odd and even factors.

Example 4. How many even factors does 2,160 have? (a) 8 (b) 24 (c) 32 (d) 40 *Solution.* $2{,}160 = 2^4 \times 3^3 \times 5$. Total = $5 \times 4 \times 2 = 40$. Odd = $4 \times 2 = 8$. Even = $40 - 8$. **Answer: (c) 32**

Type 2: factors that are multiples of k. Force the primes of k into the factor; count the remaining choices.

Example 5. How many factors of 1,080 are divisible by 6? *Solution.* $1{,}080 = 2^3 \times 3^3 \times 5$. A factor divisible by 6 needs 2 to the power 1–3 (3 choices), 3 to the power 1–3 (3 choices), 5 to the power 0–1 (2 choices). $3 \times 3 \times 2$. **Answer: 18**

- Shortcut: factors of N divisible by k = factors of N/k (here $1{,}080/6 = 180 = 2^2 \times 3^2 \times 5 \rightarrow 18$).

Type 3: perfect-square factors. Allow only even exponents.

Example 6. How many factors of 3,600 are perfect squares? *Solution.* $3{,}600 = 2^4 \times 3^2 \times 5^2$. Even exponents: $2 \rightarrow \{0, 2, 4\} = 3$; $3 \rightarrow \{0, 2\} = 2$; $5 \rightarrow \{0, 2\} = 2$. $3 \times 2 \times 2$. **Answer: 12**

Type 4: product of factors.

Example 7. Find the product of all factors of 36. *Solution.* 36 has 9 factors. Product = 36^(9/2) = 6^9. **Answer: 6^9**

Type 5: ways to write N as a product of two factors. Unordered pairs = d/2 for a non-square; (d + 1)/2 for a square (counting k × k once).

Example 8. In how many ways can 420 be written as a product of two natural numbers (order not counted)? And 144? *Solution.* 420 = $2^2 \times 3 \times 5 \times 7$ has 24 factors → 12 pairs. 144 = $2^4 \times 3^2$ has 15 factors → (15 + 1)/2 = 8. **Answer: 12; 8**

Type 6: co-prime count.

Example 9. How many numbers from 1 to 60 are co-prime to 60? *Solution.* 60 = $2^2 \times 3 \times 5$. φ = 60 × 1/2 × 2/3 × 4/5. **Answer: 16**

Example 10. Sum of the odd factors of 360. *Solution.* Odd part of 360 = $3^2 \times 5$. Sum = (1 + 3 + 9)(1 + 5) = 13 × 6. **Answer: 78**

Speed tips.

- Always factorise completely before using a formula; 3,600 is $2^4 \times 3^2 \times 5^2$, not 36 × 100.
- For "divisible by k" use the N/k shortcut; for "not divisible by k" subtract from the total.

Common traps

✗ "Factors of $2^4 \times 3^3 \times 5$ = 4 × 3 × 1 = 12." ✓ Add 1 to every exponent: 5 × 4 × 2 = **40**.

✗ "Even factors = half the total." ✓ Even = total − odd: for 2,160, 40 − 8 = **32**.

✗ "Ways to write 144 as a product of two numbers = 15/2." ✓ For a square add 1 first: (15 + 1)/2 = **8**.

✗ "Smallest number with 12 factors is 2^{11} or 72." ✓ Try every exponent pattern: **60** = $2^2 \times 3 \times 5$.

✗ "A number with exactly 3 factors can be any square." ✓ It must be the **square of a prime** (36 has 9 factors).

Quick revision

- N = $p^a q^b r^c$: factors (a+1)(b+1)(c+1).
- Sum of factors: product of (1 + p + … + p^a).
- Product of factors: N^(d/2).
- Odd factors: drop the 2s; even = total − odd.
- Factors divisible by k = factors of N/k.
- Square factors: even exponents only.
- Odd number of factors ⇔ perfect square.
- Exactly 3 factors ⇔ square of a prime.
- φ(N) = N(1 − 1/p)(1 − 1/q)…; sum of co-primes below N = Nφ(N)/2.

- Smallest with d factors: largest exponent on the smallest prime, try all patterns.

11 Remainder theorems: cyclicity, Fermat, Wilson and negative remainders

Replace each number by its remainder (or a small negative remainder), find a power that leaves 1 or −1, and the remainder of even a huge expression drops out in two lines.

Core facts

- **Division:** dividend = divisor × quotient + remainder, with 0 ≤ remainder < divisor.
- **Sum and product rules:** the remainder of a sum (or product) = the remainder of the sum (or product) of the remainders.
- **Negative remainder:** a number just below the divisor can be used as a negative: 98 on 100 is −2. Convert back by adding the divisor.
- **Cyclicity:** remainders of powers a^n on m repeat; find the first power leaving 1.
- **Fermat's little theorem:** if p is prime and p does not divide a, then **a^(p − 1) leaves 1** on p.
- **Wilson's theorem:** if p is prime, **(p − 1)! leaves p − 1** (that is, −1) on p.
- **Euler's theorem** (for any m): if a and m are co-prime, a^φ(m) leaves 1 on m (φ as in MAB-NS-10).
- **Algebra:** $a^n - b^n$ is divisible by a − b; by a + b too when n is even. $a^n + b^n$ is divisible by a + b when n is odd.

Example 1. Remainder when 47 × 53 × 61 is divided by 9. *Solution.* Remainders 2, 8, 7. 2 × 8 × 7 = 112, which leaves 4. **Answer: 4**

Example 2. Remainder when 98 × 99 is divided by 100. *Solution.* (−2) × (−1) = 2. **Answer: 2**

Example 3. Remainder when 2^{100} is divided by 7. *Solution.* $2^3 = 8$ leaves 1. $2^{100} = (2^3)^{33} \times 2$. **Answer: 2**

Explained

Type 1: Fermat for a prime divisor.

Example 4. Remainder when 5^{62} is divided by 13. (a) 1 (b) 5 (c) 12 (d) 8 *Solution.* 13 is prime, so 5^{12} leaves 1. 62 = 12 × 5 + 2, so the remainder is that of $5^2 = 25$, which is 12. **Answer: (c) 12**

- Faster: $5^2 = 25$ leaves −1, so 5^4 leaves 1; $5^{62} = (5^4)^{15} \times 5^2 \rightarrow -1$, that is 12.

Example 5. Remainder when 3^{200} is divided by 11. *Solution.* Fermat: 3^{10} leaves 1 (in fact $3^5 = 243 = 22 \times 11 + 1$ already does). 200 is a multiple of 5. **Answer: 1**

Type 2: Wilson for factorials.

Example 6. Remainders when (i) 22! and (ii) 21! are divided by 23. *Solution.* (i) 23 is prime, so 22! leaves −1, that is 22. (ii) 22! = 22 × 21!, and 22 leaves −1. So (−1) × (21!) leaves −1, meaning 21! leaves 1. **Answer: (i) 22 (ii) 1**

Type 3: negative remainders with large powers.

Example 7. Remainder when 2^{203} is divided by 17. *Solution.* $2^4 = 16$ leaves −1, so 2^8 leaves 1. 203 = 8 × 25 + 3, so the remainder is that of 2^3. **Answer: 8**

Example 8. Which of these divides $29^{36} - 1$: 28, 30, both, or neither? *Solution.* $a^n - 1$ is divisible by $a - 1 = 28$ always, and by $a + 1 = 30$ because 36 is even. **Answer: both**

Type 4: numbers formed by writing integers in a row. Use the digit sum for 3 and 9, and the last digits for 2, 4, 8.

Example 9. N = 123456789101112...3940 (1 to 40 written in a row). Find the remainders when N is divided by 9 and by 8. *Solution.* Digit sum leaves the same remainder on 9 as $1 + 2 + \ldots + 40 = 820$, and 820 leaves 1. For 8, use the last three digits, 940, which leaves 4. **Answer: 1 and 4**

Type 5: division-sum relations.

Example 10. In a division sum the divisor is 12 times the quotient and 4 times the remainder. The remainder is 18. Find the dividend. *Solution.* Divisor = 4 × 18 = 72; quotient = 72 ÷ 12 = 6. Dividend = 72 × 6 + 18. **Answer: 450**

Speed tips.

- Look for a base that is 1 more or 1 less than the divisor: 37 on 36 is 1, 35 on 36 is −1.
- For $a^n + b^n$ or $a^n - b^n$ with $a + b$ or $a - b$ equal to the divisor, the remainder is 0 without any arithmetic.

Common traps

✗ "98 × 99 on 100 leaves −2 × −1 = −2." ✓ (−2)(−1) = **+2**.

✗ "Fermat applies to any divisor: 2^8 leaves 1 on 9." ✓ Fermat needs a **prime**; on 9, 2^6 leaves 1 (Euler, φ(9) = 6), and 2^8 leaves 4.

✗ "21! on 23 leaves 22 by Wilson." ✓ Wilson gives **22!** → 22; then **21!** → **1**.

✗ "$17^{23} + 23^{23}$ on 40: work out the powers." ✓ Odd power and 17 + 23 = 40: remainder **0**.

✗ "A negative remainder is the final answer." ✓ Add the divisor: −1 on 13 is **12**.

Quick revision

- Remainder of sum/product = sum/product of remainders.
- Use negative remainders for numbers just below the divisor.
- Fermat: a^(p−1) → 1 on prime p (p ∤ a).
- Wilson: (p − 1)! → p − 1 on prime p; (p − 2)! → 1.
- Euler: a^φ(m) → 1 on m for co-prime a, m.
- $a^n - b^n \div (a - b)$ always; $\div (a + b)$ for even n.
- $a^n + b^n \div (a + b)$ for odd n.
- Last two digits = remainder on 100; $7^4 = 2{,}401$.
- Split a composite divisor into co-prime parts and combine.

- Concatenated numbers: digit sum for 3, 9; last digits for 2, 4, 8.

12 Problems on numbers: turning word statements into equations

Name the unknown, translate each phrase into symbols, and solve one or two simple equations; in MCQs, checking the options against the conditions is often faster still.

Core facts

- **Translation table:**

Words	Symbols
a number increased by 7 / exceeds y by 7	$x + 7$ / $x = y + 7$
3/5 of a number	$(3/5)x$
two numbers differ by 6	$x - y = 6$
the reciprocal of a number	$1/x$
three consecutive numbers	$x - 1, x, x + 1$

- **Sum S and difference D:** larger = $(S + D)/2$, smaller = $(S - D)/2$.
- **Sum S and product P:** $(x - y)^2 = S^2 - 4P$; $x^2 + y^2 = S^2 - 2P$.
- For three consecutive numbers, use the **middle** as x: the sum is $3x$ and the sum of squares is $3x^2 + 2$.
- **Fractions:** call it n/d; each condition gives one equation.
- Dividend = divisor × quotient + remainder.

Example 1. A number added to 3/5 of itself gives 64. Find the number. *Solution.* $(8/5)x = 64$. **Answer: 40**

Example 2. Two numbers add to 50 and differ by 14. Find them. *Solution.* $(50 + 14)/2 = 32$ and $(50 - 14)/2 = 18$. **Answer: 32 and 18**

Example 3. The difference between a number and two-fifths of it is 36. Find the number. *Solution.* $(3/5)x = 36$. **Answer: 60**

Explained

Type 1: sum and product given. Use the identities instead of solving for each number.

Example 4. Two numbers have sum 17 and product 72. Find (i) their difference and (ii) the sum of their squares. (a) 1, 145 (b) 1, 217 (c) 3, 145 (d) 7, 217 *Solution.* $(x - y)^2 = 289 - 288 = 1$, so the difference is 1. $x^2 + y^2 = 289 - 144 = 145$. (The numbers are 8 and 9.) **Answer: (a) 1, 145**

Type 2: number and its reciprocal.

Example 5. A number plus its reciprocal is 26/5. Find the number. *Solution.* $x + 1/x = 5 + 1/5$, so $x = 5$ or $1/5$. **Answer: 5 or 1/5**

Type 3: fraction problems.

Example 6. If 1 is added to both the numerator and denominator of a fraction, it becomes 2/3. If 1 is subtracted from both, it becomes 1/2. Find the fraction. *Solution.* $3(n + 1) = 2(d + 1)$ and $2(n - 1) = d - 1$. The second gives $d = 2n - 1$; substitute: $3n + 3 = 4n$, so $n = 3$, $d = 5$. Check: $4/6 = 2/3$ and $2/4 = 1/2$. **Answer: 3/5**

Type 4: wrong operation. Set (correct result) − (wrong result) = the stated gap.

Example 7. A student multiplied a number by 3/5 instead of 5/3 and got an answer 32 less than the correct one. Find the number. *Solution.* $(5/3 - 3/5)x = (16/15)x = 32$. **Answer: 30**

Type 5: consecutive numbers.

Example 8. The squares of three consecutive natural numbers add to 434. Find the numbers. *Solution.* With middle x: $3x^2 + 2 = 434$, so $x^2 = 144$, $x = 12$. **Answer: 11, 12, 13**

Type 6: dividing a number into parts.

Example 9. Divide 75 into two parts so that one-third of the first plus one-fifth of the second is 19. *Solution.* $x/3 + (75 - x)/5 = 19$. Multiply by 15: $5x + 225 - 3x = 285$, so $x = 30$. **Answer: 30 and 45**

Type 7: quotient and remainder statements.

Example 10. Three times the larger of two numbers, divided by the smaller, gives quotient 5 and remainder 4. The numbers differ by 26. Find them. *Solution.* $3L = 5S + 4$ and $L = S + 26$. So $3S + 78 = 5S + 4$, $S = 37$, $L = 63$. Check: $189 = 5 \times 37 + 4$. **Answer: 63 and 37**

Speed tips.

- In MCQs, test the options in the condition that is easiest to check (often the sum or difference) before writing any equation.
- Clear fractions at once by multiplying through by the LCM of the denominators.

Common traps

✗ "Three consecutive numbers: x, x + 1, x + 2, so the sum of squares is messy." ✓ Use the **middle** as x: sum of squares = $\mathbf{3x^2 + 2}$.

✗ "x + 1/x = 26/5 gives only x = 5." ✓ The reciprocal **1/5** also works; check the options for both.

✗ "Multiplied by 3/5 instead of 5/3: difference = (5/3 − 3/5) × 32." ✓ The gap equals (16/15)x, so x = 32 × 15/16 = **30**.

✗ "Sum 17, product 72: sum of squares = $17^2 - 72$." ✓ Subtract **2P**: 289 − 144 = **145**.

✗ "B says 41 − 15 = 26." ✓ The pair adds to 41 + **1**: B says **27**.

Quick revision

- Name the unknown first; one condition, one equation.
- Larger = (S + D)/2; smaller = (S − D)/2.

- $(x - y)^2 = S^2 - 4P$; $x^2 + y^2 = S^2 - 2P$.
- Middle of three consecutive numbers as x; squares sum to $3x^2 + 2$.
- Fraction n/d: two conditions, two equations.
- Wrong operation: correct − wrong = gap.
- Ratio of sum : difference gives the numbers as multiples of k.
- Counting both ways: the pair adds to (last + first).
- Check the answer in every condition of the question.

13 Speed maths: multiplication, squaring and percentage shortcuts

Most aptitude arithmetic is a handful of patterns: turn 5, 25 and 125 into divisions, 9s into subtractions, nearby numbers into a base, and percentages into fractions.

Core facts

- **× 5 = × 10 ÷ 2; × 25 = × 100 ÷ 4; × 125 = × 1,000 ÷ 8.**
- **× 9, 99, 999:** N × 999 = N × 1,000 − N.
- **× 11 (two-digit ab):** write a, (a + b), b, carrying if a + b ≥ 10: 63 × 11 = 693; 78 × 11 = 7(15)8 = 858.
- **Square of a number ending in 5:** $n5^2 = n \times (n + 1)$, then write 25: $85^2 = 72|25$.
- **Same tens digit, units adding to 10:** 64 × 66 = (6 × 7)|(4 × 6) = 4,224.
- **Difference of squares:** $(a + b)(a - b) = a^2 - b^2$: $73 \times 67 = 70^2 - 3^2$.
- **Fraction–percent table:**

Fraction	Percent	Fraction	Percent
1/6	16.67%	1/11	9.09%
1/7	14.29%	1/12	8.33%
1/8	12.5%	1/16	6.25%
1/9	11.11%	3/8	37.5%

- **x% of y = y% of x.**

Example 1. Find 4,628 × 25. *Solution.* 4,62,800 ÷ 4. **Answer: 1,15,700**

Example 2. Find 85^2. *Solution.* 8 × 9 = 72, then 25. **Answer: 7,225**

Example 3. Find 78 × 11. *Solution.* 7, (7 + 8 = 15), 8: carry 1 to get 858. **Answer: 858**

Explained

Type 1: multiplying by 9s.

Example 4. Find 7,346 × 999. *Solution.* 73,46,000 − 7,346. **Answer: 73,38,654**

Type 2: squaring near a base. $(\text{Base} \pm d)^2 = \text{base}^2 \pm 2 \times \text{base} \times d + d^2$. For base 100: $(100 - d)^2$ is $(100 - 2d)$ followed by d^2 as two digits.

Example 5. Find (i) 97^2, (ii) 104^2, (iii) 996^2. *Solution.* (i) 97 is 3 below 100: 94|09. (ii) 104 is 4 above: 108|16. (iii) Base 1,000, 4 below: 992|016. **Answer: (i) 9,409 (ii) 10,816 (iii) 9,92,016**

Type 3: same tens digit, units summing to 10.

Example 6. Find 64 × 66. *Solution.* 6 × 7 = 42; 4 × 6 = 24. **Answer: 4,224**

Type 4: difference of squares. Use it when the two numbers are equally spaced around a round number.

Example 7. Find 73 × 67. (a) 4,881 (b) 4,891 (c) 4,901 (d) 4,991 *Solution.* $70^2 - 3^2 = 4{,}900 - 9$. **Answer: (b) 4,891**

Type 5: percentage shortcuts.

Example 8. Find (i) 18% of 450, (ii) 12.5% of 968, (iii) 16⅔% of 4,326. *Solution.* (i) Swap: 18% of 450 = 45% of 180 = 90 − 9 = 81. (ii) 1/8 of 968 = 121. (iii) 1/6 of 4,326 = 721. **Answer: (i) 81 (ii) 121 (iii) 721**

Example 9. Find 4,321 × 125. *Solution.* 43,21,000 ÷ 8 = 5,40,125. **Answer: 5,40,125**

Example 10. Find 46 × 25 × 4. *Solution.* Pair 25 × 4 = 100 first. **Answer: 4,600**

Speed tips.

- Rearrange products to pair 2 × 5, 4 × 25 or 8 × 125 before multiplying.
- Write Indian commas (lakhs, crores) as you go; misplaced zeros cause most errors.

Common traps

✗ "× 25 means × 100 × 4." ✓ It is × 100 **÷ 4**.

✗ "97^2 = 94|9 = 949." ✓ The right part needs two digits for base 100: 94|**09** = 9,409.

✗ "+20% then −20% leaves the price unchanged." ✓ Net change is **−4%**.

✗ "78 × 11 = 7158." ✓ Carry the 1 from 15: **858**.

✗ "Digit roots match, so the product is right." ✓ Also check the **last two digits**; digit roots miss swapped digits.

Quick revision

- × 5: ÷ 2 with a 0; × 25: ÷ 4 with 00; × 125: ÷ 8 with 000.
- $N \times 99\ldots9 = N \times 10^k - N$.
- ab × 11 = a | a + b | b.
- $n5^2 = n(n + 1) \mid 25$.
- Same tens, units adding to 10: a(a + 1) | product of units.
- $(100 \pm d)^2 = (100 \pm 2d) \mid d^2$.
- $(a + b)(a - b) = a^2 - b^2$.
- 1/6 = 16.67%, 1/7 = 14.29%, 1/8 = 12.5%, 1/12 = 8.33%.
- Successive change: a + b + ab/100.
- Check products by digit root and the last two digits.

14 Place value, Roman numerals and counting digits in a range

Read every digit by its place, write Roman numerals from the seven symbols and six subtractive pairs, and count digits place by place or block by block, never number by number.

Core facts

- **Indian system:** ones, tens, hundreds, thousands, ten thousands, **lakhs**, ten lakhs, **crores**, ten crores. Commas: 4,52,46,317.
- **International system:** thousands, millions, billions. Commas every three digits: 45,246,317.
- **1 lakh = 100 thousand; 10 lakh = 1 million; 1 crore = 10 million; 100 crore = 1 billion.**
- **Place value** = digit × its place; **face value** = the digit. Place value of 0 is 0.
- **Roman numerals:** I = 1, V = 5, X = 10, L = 50, C = 100, D = 500, M = 1,000.
  - **Subtractive pairs only:** IV (4), IX (9), XL (40), XC (90), CD (400), CM (900).
  - I, X, C, M may repeat up to **three** times; **V, L, D never repeat** and are never subtracted.
  - There is no Roman symbol for zero.
- **Digits used in writing numbers:** 1–9 use **9**; 10–99 use **180**; 100–999 use **2,700**. So 1–99 use 189 and 1–999 use 2,889.

Example 1. Place value of 8 in 3,87,45,612. *Solution.* 8 is in the ten-lakh place: 8 × 10,00,000. **Answer: 80,00,000 (8 million)**

Example 2. Write MCMXCIV in numerals. *Solution.* M = 1,000; CM = 900; XC = 90; IV = 4. **Answer: 1,994**

Example 3. How many digits are used to write the numbers 1 to 60? *Solution.* 9 single-digit numbers, then 51 two-digit numbers (10–60): 9 + 102. **Answer: 111**

Explained

Type 1: place values of repeated digits.

Example 4. Difference between the place values of the two 4s in 4,52,46,317. (a) 3,99,60,000 (b) 4,00,00,000 (c) 3,99,96,000 (d) 39,960 *Solution.* The first 4 is in the crore place: 4,00,00,000. The second is in the ten-thousand place: 40,000. Difference = 3,99,60,000. **Answer: (a) 3,99,60,000**

Type 2: number names in both systems.

Example 5. Write "seven crore four lakh thirty thousand nine" in figures, and read it in the international system. *Solution.* 7,04,30,009 = 70,430,009, read "seventy million four hundred thirty thousand nine". **Answer: 7,04,30,009**

Type 3: writing Roman numerals. Break the number into thousands, hundreds, tens and units, and write each part.

Example 6. Write 2,749 in Roman numerals. *Solution.* 2,000 = MM; 700 = DCC; 40 = XL; 9 = IX. **Answer: MMDCCXLIX**

Type 4: pages from digits used.

Example 7. 1,002 digits are used to number the pages of a book from page 1. How many pages does it have? *Solution.* Pages 1–99 use 189 digits. The remaining 813 digits give 813 ÷ 3 = 271 three-digit pages (100 to 370). **Answer: 370**

Example 8. How many digits are used to write 1 to 1,000? *Solution.* 9 + 180 + 2,700 + 4. **Answer: 2,893**

Type 5: how many times a digit is written. Count each place separately.

Example 9. How many times is the digit 5 written from 1 to 500? How many numbers contain at least one 5? *Solution.* Units place: 5, 15, ..., 495 → 50. Tens place: 50–59 in each of the 5 hundreds → 50. Hundreds place: only 500 → 1. Total 101. For numbers containing a 5, count 0–499 with no 5: 5 × 9 × 9 = 405, which includes 0, so 404 in 1–499. That leaves 499 − 404 = 95 containing a 5, plus 500 itself. **Answer: 101 times; 96 numbers**

Type 6: complement counting. "Contains at least one" = total − "contains none".

Example 10. How many numbers from 1 to 400 contain the digit 4 at least once? *Solution.* From 0 to 399, numbers with no 4: hundreds 0–3 (4 choices) × 9 × 9 = 324, which includes 0, so 323 in 1–399. 400 contains a 4. So 1–400 has 400 − 323 = 77 containing a 4. **Answer: 77**

Speed tips.

- "How many times is the digit written" counts 55 twice; "how many numbers contain it" counts 55 once.
- For page problems, subtract 189 (pages 1–99) before dividing by 3.

Common traps

✗ "1 million = 1 crore." ✓ **1 million = 10 lakh**; 1 crore = 10 million.

✗ "1,999 = MIM." ✓ Subtract only within the six pairs: **MCMXCIX**.

✗ "4 = IIII and 40 = XXXX." ✓ No symbol repeats four times: **IV** and **XL**.

✗ "1,002 digits ÷ 3 = 334 pages." ✓ Pages 1–99 use only 189 digits: **370** pages.

✗ "Digit 5 appears 96 times from 1 to 500." ✓ 96 is how many **numbers contain** 5; the digit is written **101** times.

✗ "Numbers with no 4 from 1 to 399 = 4 × 9 × 9 = 324." ✓ That counts 0 too: **323**.

Quick revision

- 1 lakh = 100,000; 10 lakh = 1 million; 1 crore = 10 million; 100 crore = 1 billion.
- Place value = digit × place; face value = digit.

- Roman: I 1, V 5, X 10, L 50, C 100, D 500, M 1,000.
- Subtractive pairs: IV, IX, XL, XC, CD, CM only.
- V, L, D never repeat; I, X, C, M at most three times.
- Digits: 9 (1–9), 180 (10–99), 2,700 (100–999).
- Pages from digits: (digits − 189)/3 + 99.
- Count a digit place by place; numbers containing it by complement.
- Sum of digits 1–99 = 900; 1–999 = 13,500.