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STATISTICS

Probability and probability distributions

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Chapter 1 · Statistics

Probability and probability distributions

3 TOPICS

1.1 Random variables, probability mass and density functions, mathematical expectation and variance

A random variable gives a number to every outcome of an experiment; its probabilities are described by a mass function (discrete) or a density function (continuous), and its expectation E(X) and variance V(X) are the probability-weighted mean and spread.

CORE FACTS

- **Random variable (r.v.) X:** a real-valued function defined on the sample space.
- **Discrete r.v.:** takes countable values (number of heads, number of defects).
- **Continuous r.v.:** takes any value in an interval (height, time, weight).
- **Probability mass function (pmf)** p(x) = P(X = x): **$p(x) \geq 0$** and **$\Sigma\, p(x) = 1$**.
- **Probability density function (pdf)** f(x): **$f(x) \geq 0$** and **$\int f(x)\, dx = 1$** over the whole range.
- Continuous case: P(a ≤ X ≤ b) = ∫ from a to b of f(x) dx, and **P(X = a) = 0** for every single point a.
- **Distribution function** F(x) = P(X ≤ x). It is **non-decreasing**, F(−∞) = 0, F(+∞) = 1, and it is right-continuous.
- Discrete: F is a **step function**. Continuous: **$f(x) = dF(x)/dx$**.
- **Expectation:** E(X) = **$\Sigma\, x\, p(x)$** (discrete) or **$\int x\, f(x)\, dx$** (continuous).
- **Variance:** V(X) = E[X − E(X)]² = **$E(X^2) - [E(X)]^2$**. SD = √V(X).
- **$E(aX + b) = aE(X) + b$**; **$V(aX + b) = a^2\, V(X)$**. Change of origin does not change variance.
- E(constant) = constant; V(constant) = 0.
- **Addition theorem:** E(X + Y) = E(X) + E(Y), for **any** X and Y.
- **Multiplication theorem:** E(XY) = E(X) E(Y), when X and Y are **independent**.
- **Covariance:** Cov(X, Y) = E(XY) − E(X)E(Y). Independent ⇒ Cov = 0.
- **$V(X \pm Y) = V(X) + V(Y) \pm 2\, Cov(X, Y)$**; for independent X, Y: V(X ± Y) = **$V(X) + V(Y)$** (plus in both cases).

- **Moment generating function:** M_X(t) = E(e^(tX)). The r-th raw moment is the coefficient of t^r/r! in it.

1. Discrete case.

Example 1. A fair die is thrown. Find E(X) and V(X), where X is the number shown. *Solution.* E(X) = (1 + 2 + … + 6)/6 = 21/6 = 7/2. E(X²) = (1 + 4 + 9 + 16 + 25 + 36)/6 = 91/6. V(X) = 91/6 − 49/4 = 182/12 − 147/12. **Answer: E(X) = 3.5; V(X) = 35/12**

Example 2. X takes the values 1, 2, 3, 4 with p(x) = kx. Find k, E(X), V(X) and F(2). *Solution.* Σ p(x) = k(1 + 2 + 3 + 4) = 10k = 1, so k = 1/10. E(X) = Σ x · x/10 = (1 + 4 + 9 + 16)/10 = 3. E(X²) = (1 + 8 + 27 + 64)/10 = 10, so V(X) = 10 − 9 = 1. F(2) = p(1) + p(2) = 3/10. **Answer: k = 1/10; E(X) = 3; V(X) = 1; F(2) = 0.3**

Example 3. A player wins ₹10 if a coin shows head and loses ₹5 if it shows tail. Find the expected gain. *Solution.* E = 10 × 1/2 + (−5) × 1/2 = 5 − 2.5. **Answer: ₹2.50 per game** A game is called **fair** when the expected gain is 0.

2. Continuous case. Probability is an **area** under f(x), not the height of f(x). The value f(x) itself can be more than 1.

Example 4. f(x) = 2x for 0 < x < 1 (and 0 elsewhere). Check that it is a pdf. Find P(X ≤ 1/2), E(X) and V(X). *Solution.* $\int_0^1 2x\,dx = [x^2]_0^1 = 1$, and f ≥ 0, so it is a pdf. P(X ≤ 1/2) = [x²] from 0 to 1/2 = 1/4. E(X) = $\int_0^1 2x^2\,dx = 2/3$. E(X²) = $\int_0^1 2x^3\,dx = 1/2$. V(X) = 1/2 − 4/9 = 1/18. **Answer: P = 1/4; E(X) = 2/3; V(X) = 1/18**

3. Linear changes.

Example 5. E(X) = 5 and V(X) = 4. Find E(3X − 2), V(3X − 2) and the SD of 3X − 2. *Solution.* E = 3 × 5 − 2 = 13. V = 3² × 4 = 36 (the −2 has no effect). SD = 6. **Answer: 13, 36, 6**

Example 6. X and Y are independent with V(X) = 4 and V(Y) = 9. Find V(X − Y). *Solution.* For independent variables the variances **add**, even for a difference. 4 + 9. **Answer: 13**

Deeper

1. Discrete versus continuous.

Point	Discrete	Continuous
Function	pmf p(x) = P(X = x)	pdf f(x); P(X = x) = 0
Total	Σ p(x) = 1	∫ f(x) dx = 1
E(X)	Σ x p(x)	∫ x f(x) dx
F(x)	step function	continuous; F′(x) = f(x)

Because $P(X = a) = 0$ for a continuous r.v., $P(a < X < b) = P(a \le X \le b)$. For a discrete r.v. the end points matter.

2. Why $V(X) = E(X^2) - [E(X)]^2$. Write $\mu = E(X)$. Then $E(X - \mu)^2 = E(X^2) - 2\mu E(X) + \mu^2 = E(X^2) - \mu^2$. Since $V(X) \ge 0$, we always have **$E(X^2) \ge [E(X)]^2$**.

3. Why $V(aX + b) = a^2V(X)$. $aX + b - E(aX + b) = a(X - \mu)$. Squaring gives $a^2(X - \mu)^2$, and its expectation is $a^2V(X)$. The constant b cancels: **variance is independent of the change of origin but not of scale.**

4. Covariance cautions.

- Independent ⇒ $Cov(X, Y) = 0$. The converse is **false**: zero covariance does not prove independence.
- Example: X takes −1, 0, 1 each with probability 1/3, and $Y = X^2$. Then $E(X) = 0$ and $E(XY) = E(X^3) = 0$, so Cov = 0. But Y is fully decided by X.
- $Cov(aX, bY) = ab\ Cov(X, Y)$. $Cov(X, X) = V(X)$.

5. Number of heads in three tosses. X = 0, 1, 2, 3 with probabilities 1/8, 3/8, 3/8, 1/8. $E(X) = 12/8 = 3/2$. $E(X^2) = 24/8 = 3$. $V(X) = 3 - 9/4 = 3/4$. This matches the binomial results np and npq (STA-PD-03).

6. Moment generating function (mgf).

- $M_X(t) = E(e^{tX}) = 1 + t\mu'_1 + (t^2/2!)\mu'_2 + \ldots$
- μ'_r = r-th derivative of $M_X(t)$ at $t = 0$.
- $M_{(aX+b)}(t) = e^{bt} M_X(at)$.
- For **independent** X and Y, $M_{(X+Y)}(t) = M_X(t) M_Y(t)$. This is the key tool for additive properties.
- The mgf, when it exists, decides the distribution uniquely.

7. Chebyshev's inequality (a moment inequality). For any r.v. with mean μ and SD σ, and any $k > 0$: **$P(|X - \mu| < k\sigma) \ge 1 - 1/k^2$**, or $P(|X - \mu| \ge k\sigma) \le 1/k^2$.

- $k = 2$: at least **3/4** of the probability lies within $\mu \pm 2\sigma$.
- $k = 3$: at least 8/9 lies within $\mu \pm 3\sigma$.
- It holds for **every** distribution, so it is a rough bound. For the normal curve the true figures are much higher (95.45% and 99.73%).

COMMON TRAPS

- ✗ "For a continuous r.v., P(X = 2) = f(2)." ✓ P(X = 2) = **0**; f(2) is a density, not a probability.
- ✗ "A pdf value cannot exceed 1." ✓ It can (f(x) = 2x gives f(0.9) = 1.8); only the **total area** must be 1.
- ✗ "V(aX + b) = aV(X) + b." ✓ **V(aX + b) = a²V(X)**.
- ✗ "V(X − Y) = V(X) − V(Y) for independent X, Y." ✓ **V(X) + V(Y)**.
- ✗ "E(XY) = E(X)E(Y) always." ✓ Only for **independent** (more precisely, uncorrelated) X and Y. E(X + Y) = E(X) + E(Y) is always true.
- ✗ "Cov(X, Y) = 0 means X and Y are independent." ✓ Zero covariance does **not** imply independence.
- ✗ "V(X) = E(X²) − E(X)." ✓ V(X) = E(X²) − **[E(X)]²**.
- ✗ "F(x) can decrease." ✓ A distribution function is **non-decreasing**, from 0 to 1.

1.2 Bernoulli, binomial and Poisson distributions

A Bernoulli trial has two outcomes; the binomial B(n, p) counts successes in n independent trials (mean np, variance npq), and the Poisson P(λ) counts rare events (mean = variance = λ) and is the limit of the binomial when n is large and p is small.

CORE FACTS

- **Bernoulli trial:** only two outcomes, success (probability p) or failure (probability **q = 1 − p**).
- **Bernoulli distribution:** p(x) = p^x q^(1−x), x = 0, 1. **Mean p, variance pq.** (James Bernoulli)
- **Binomial distribution X ~ B(n, p):** p(x) = **^nC_x p^x q^(n−x)**, x = 0, 1, …, n.
- Conditions: **n fixed**, trials **independent**, two outcomes each, **p the same** in every trial.
- Binomial **mean = np**, **variance = npq**, so **variance < mean** (q < 1).
- Binomial $\mu_3 = npq(q - p)$; $\beta_1 = (q - p)^2/(npq)$; $\beta_2 = 3 + (1 - 6pq)/(npq)$.
- **Symmetric when p = q = 1/2**; positive skew if p < 1/2; negative skew if p > 1/2.
- Binomial mgf: **(q + pe^t)^n**.
- **Poisson distribution X ~ P(λ):** p(x) = **e^(−λ) λ^x / x!**, x = 0, 1, 2, … (S.D. Poisson, 1837).
- Poisson **mean = variance = λ**; $\mu_3 = \lambda$; $\beta_1 = 1/\lambda$; $\beta_2 = 3 + 1/\lambda$.

- Poisson is **always positively skewed** and leptokurtic; it tends to symmetry as λ grows.
- Poisson mgf: **e^(λ(e^t − 1))**.
- **Poisson as a limit of the binomial:** n → ∞, p → 0, with **np = λ** fixed.
- **Additive property:** sum of independent $B(n_1, p)$ and $B(n_2, p)$ is $B(n_1 + n_2, p)$ (same p); sum of independent $P(\lambda_1)$ and $P(\lambda_2)$ is **$P(\lambda_1 + \lambda_2)$**.

1. Using the binomial formula. Find n and p, then count the ways (${}^nC_x$) and multiply.

Example 1. A fair coin is tossed 4 times. Find P(exactly 2 heads) and P(at least one head). *Solution.* n = 4, p = q = 1/2. $P(X = 2) = {}^4C_2\,(1/2)^4 = 6/16 = 3/8$. $P(X \geq 1) = 1 - P(X = 0) = 1 - 1/16$. **Answer: 3/8; 15/16**

Example 2. A binomial variable has mean 4 and variance 3. Find n and p. *Solution.* q = npq/np = 3/4, so p = 1/4. n = 4/(1/4) = 16. **Answer: n = 16, p = 1/4**

A question that says "binomial with mean 3 and variance 4" has **no** answer: it would need q = 4/3 > 1.

Example 3. Three coins are tossed 80 times. Find the expected (theoretical) frequencies of 0, 1, 2, 3 heads. *Solution.* Expected frequency = $N \times {}^3C_x\,(1/2)^3 = 80 \times (1, 3, 3, 1)/8$. **Answer: 10, 30, 30, 10** This is how a binomial distribution is **fitted** to data: estimate p from the mean (p = x̄/n), then multiply N by each probability.

2. Using the Poisson formula. Poisson fits counts of **rare** events in a fixed time, space or lot: misprints per page, accidents per day, defects per item, telephone calls per minute, deaths by horse-kick (Bortkiewicz's classic data).

Example 4. Accidents at a junction follow a Poisson law with mean 2 per week. Find P(no accident) and P(at least one accident) in a week. Take e^(−2) = 0.1353. *Solution.* P(0) = e^(−2) = 0.1353. $P(X \geq 1) = 1 - 0.1353$. **Answer: 0.1353; 0.8647**

Example 5. For a Poisson variable, P(X = 1) = P(X = 2). Find the mean. *Solution.* $e^{-\lambda}\lambda = e^{-\lambda}\lambda^2/2$ gives λ = 2. **Answer: 2** (In the same way, P(X = 2) = P(X = 3) gives λ = 3.)

Example 6. 2% of the bulbs made by a firm are defective. Use the Poisson approximation to find the chance that a box of 100 has no defective bulb. *Solution.* λ = np = 100 × 0.02 = 2. P(0) = e^(−2) ≈ 0.135. (The exact binomial value $0.98^{100} \approx 0.133$ is very close.) **Answer: about 0.135**

Deeper

1. Comparison.

Point	Bernoulli	Binomial B(n, p)	Poisson P(λ)
Values	0, 1	0, 1, ..., n	0, 1, 2, ... (no upper limit)
Mean	p	np	λ
Variance	pq	npq (< mean)	λ (= mean)
Parameters	p	n, p	λ only

Mean > variance suggests binomial. Mean = variance suggests Poisson. (Mean < variance points to the negative binomial, not covered here.)

2. Mean and variance of the binomial (sketch). X is the sum of n independent Bernoulli variables. Each has mean p and variance pq. By the addition theorem E(X) = np. Because the trials are independent, the variances add: V(X) = npq. The largest variance is **n/4**, at p = 1/2.

3. Recurrence relations (used to build tables quickly).

- Binomial: **p(x + 1) = [(n − x)/(x + 1)] (p/q) p(x)**.
- Poisson: **p(x + 1) = [λ/(x + 1)] p(x)**.

Start from p(0) = q^n or e^(−λ), then step upward.

4. Mode.

- Binomial: look at (n + 1)p.
  - Not an integer: mode = its **integral part**. B(10, 1/2): (n + 1)p = 5.5, so the mode is **5**.
  - An integer m: **bimodal**, modes at **m − 1 and m**. B(5, 1/3): (n + 1)p = 2, modes **1 and 2** (both have probability 80/243).
- Poisson: λ not an integer: mode = integral part of λ. λ an integer: bimodal at **λ − 1 and λ**.

5. Derivation of the Poisson limit (sketch). Put p = λ/n in ${}^{n}C_x\, p^x q^{(n-x)}$. As n → ∞, n(n − 1)…(n − x + 1)/n^x → 1, $(1 - \lambda/n)^{(-x)}$ → 1, and $(1 - \lambda/n)^n \to e^{(-\lambda)}$. What is left is $e^{(-\lambda)}\, \lambda^x/x!$.

6. Moments of the Poisson (sketch). $\Sigma\, x\, e^{(-\lambda)}\, \lambda^x/x! = \lambda\, e^{(-\lambda)}\, \Sigma\, \lambda^{(x-1)}/(x-1)! = \lambda\, e^{(-\lambda)}\, e^{\lambda} = \lambda$. In the same way $E[X(X-1)] = \lambda^2$, so $E(X^2) = \lambda^2 + \lambda$ and $V(X) = \lambda$.

7. Limits to the normal. The binomial tends to the normal when n is large and p is not close to 0 or 1. The Poisson tends to the normal as λ → ∞ (since $\beta_1 = 1/\lambda \to 0$ and $\beta_2 \to 3$). See STA-PD-04.

Example 7. Six fair coins are tossed. Find P(at least 5 heads). *Solution.* $({}^{6}C_5 + {}^{6}C_6)/2^6 = (6 + 1)/64$. **Answer: 7/64**

COMMON TRAPS

- ✗ "Binomial variance = np." ✓ Variance = **npq**; mean = np.
- ✗ "A binomial can have mean 3 and variance 4." ✓ Impossible: binomial variance is always **less than** the mean.
- ✗ "Poisson mean = λ, variance = √λ." ✓ Variance = **λ**; SD = √λ.
- ✗ "The binomial is always symmetric." ✓ Only when **p = 1/2**.
- ✗ "The Poisson can be negatively skewed." ✓ $\beta_1 = 1/\lambda$ with $\mu_3 = \lambda > 0$: **always positively skewed**.
- ✗ "Poisson limit: n → ∞ and p → 1." ✓ **p → 0**, with np = λ finite.
- ✗ "Any two binomials add to a binomial." ✓ Only when they have the **same p** (and are independent).
- ✗ "If (n + 1)p is an integer m, the mode is m." ✓ The distribution is **bimodal**: m − 1 and m.
- ✗ "In a binomial the trials may have different p." ✓ p must be **constant** across trials.

1.3 Normal distribution: properties, standard normal, area rules and uses

The normal distribution $N(\mu, \sigma^2)$ is the symmetric bell-shaped continuous distribution fixed by its mean μ and variance σ^2; every normal question is solved by changing X to the standard normal $Z = (X - \mu)/\sigma$ and reading areas from the Z table.

CORE FACTS

- The shape, moments and β values of the normal curve are also in the skewness note (STA-DK-02). This note is about **using** the distribution.
- **pdf:** $f(x) = [1/(\sigma\sqrt{(2\pi)})]\, e^{-(x-\mu)^2/(2\sigma^2)}$, $-\infty < x < \infty$.
- **Notation $X \sim N(\mu, \sigma^2)$:** the second parameter is the **variance**, not the SD.
- Found by **De Moivre (1733)** as a limit of the binomial; developed by **Laplace and Gauss** (hence "Gaussian").
- **Mean = median = mode = μ.** Symmetric about $x = \mu$. Skewness $\beta_1 = 0$; kurtosis $\beta_2 = 3$ (mesokurtic).
- **Points of inflexion: $x = \mu \pm \sigma$.** The curve never touches the x-axis (asymptotic).
- **Maximum ordinate** at $x = \mu$: $1/(\sigma\sqrt{(2\pi)}) \approx 0.3989/\sigma$.
- **Quartiles:** $\mu \pm 0.6745\sigma$. **QD ≈ (2/3)σ**, **MD ≈ (4/5)σ** (exactly $\sigma\sqrt{(2/\pi)} \approx 0.7979\sigma$).
- So **QD : MD : SD ≈ 10 : 12 : 15**.
- **Standard normal variate $Z = (X - \mu)/\sigma \sim N(0, 1)$**; pdf $\varphi(z) = (1/\sqrt{(2\pi)})\, e^{-z^2/2}$.
- **Area rules:** $\mu \pm \sigma$ → **68.27%**; $\mu \pm 2\sigma$ → **95.45%**; $\mu \pm 3\sigma$ → **99.73%**.
- **$\mu \pm 1.96\sigma$ → 95%**; **$\mu \pm 2.58\sigma$ → 99%**; $\mu \pm 1.645\sigma$ → 90%.
- **mgf:** $M_X(t) = e^{(\mu t + \sigma^2 t^2/2)}$.
- **Additive property:** if $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent, then **$X \pm Y \sim N(\mu_1 \pm \mu_2, \sigma_1^2 + \sigma_2^2)$**.
- **$aX + b \sim N(a\mu + b, a^2\sigma^2)$.**

1. The three-step method.

1. Convert: $z = (x - \mu)/\sigma$.
2. Read the table area from 0 to $|z|$.
3. Combine with **0.5** (each half of the curve has area 0.5) and use symmetry.

Table values used below (area from 0 to z): 0.5 → 0.1915; 1 → 0.3413; 1.4 → 0.4192; 1.5 → 0.4332; 1.9 → 0.4713; 2 → 0.4772; 3 → 0.4987.

Rules of thumb.

- $P(Z > z) = 0.5 - \text{area}(0 \text{ to } z)$.
- $P(Z < z) = 0.5 + \text{area}(0 \text{ to } z)$, for $z > 0$.
- $P(-a < Z < b) = \text{area}(0 \text{ to } a) + \text{area}(0 \text{ to } b)$.
- $P(a < Z < b)$, both positive = $\text{area}(0 \text{ to } b) - \text{area}(0 \text{ to } a)$.

- Using the cumulative form $\Phi(z) = P(Z \le z)$: **$\Phi(-z) = 1 - \Phi(z)$**.

Example 1. $X \sim N(50, 100)$. Find $P(X > 70)$ and $P(40 < X < 60)$. *Solution.* $\sigma = \sqrt{100} = 10$. For 70: $z = (70 - 50)/10 = 2$. $P(Z > 2) = 0.5 - 0.4772 = 0.0228$. For 40 and 60: $z = -1$ and 1. Area $= 2 \times 0.3413 = 0.6826$. **Answer: 0.0228; 0.6826**

Example 2. Marks of 1,000 students are normal with mean 50 and SD 10. How many score between 45 and 60? *Solution.* $z = -0.5$ and $z = 1$. Area $= 0.1915 + 0.3413 = 0.5328$. $0.5328 \times 1{,}000 = 532.8$. **Answer: about 533 students**

Example 3. $X \sim N(30, 25)$. Find $P(X < 20)$. *Solution.* $\sigma = 5$. $z = (20 - 30)/5 = -2$. $P(Z < -2) = P(Z > 2) = 0.5 - 0.4772$. **Answer: 0.0228**

2. Working backwards: finding μ and σ.

Example 4. In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find μ and σ. (Area 0 to 0.5 = 0.19; area 0 to 1.4 = 0.42.) *Solution.*

- 31% below 45 means area 0.19 between 45 and μ, so 45 is $z = -0.5$: $45 = \mu - 0.5\sigma$.
- 8% above 64 means area 0.42 between μ and 64, so 64 is $z = 1.4$: $64 = \mu + 1.4\sigma$.
- Subtract: $19 = 1.9\sigma$, so $\sigma = 10$ and $\mu = 50$.

Answer: μ = 50, σ = 10

Deeper

1. Sums and differences of normal variables.

Example 5. $X \sim N(10, 9)$ and $Y \sim N(20, 16)$ are independent. Find the distribution of $X + Y$ and of $Y - X$. *Solution.* $X + Y \sim N(30, 9 + 16) = N(30, 25)$, SD 5. $Y - X \sim N(10, 25)$, also SD 5: the variances add even for a difference. **Answer: N(30, 25) and N(10, 25)**

Consequence: the mean $\bar{x}$ of n independent $N(\mu, \sigma^2)$ values is exactly **$N(\mu, \sigma^2/n)$**. Its SD, $\sigma/\sqrt{n}$, is called the **standard error**.

2. Normal approximation to the binomial and Poisson.

- Binomial $B(n, p) \approx N(np, npq)$ when n is large and p is not near 0 or 1 (a working rule: np and nq both at least 5).
- Poisson $P(\lambda) \approx N(\lambda, \lambda)$ when λ is large.
- **Continuity correction:** a discrete count is spread over an interval of width 1. So $P(X \ge 60)$ uses 59.5; $P(X \le 40)$ uses 40.5; $P(X = 50)$ uses 49.5 to 50.5.

Example 6. A fair coin is tossed 100 times. Find (approximately) P(at least 60 heads). (Area 0 to 1.9 = 0.4713.) *Solution.* $\mu = np = 50$, $\sigma = \sqrt{(npq)} = \sqrt{25} = 5$. With continuity correction, $z = (59.5 - 50)/5 = 1.9$. $P = 0.5 - 0.4713 = 0.0287$. (The exact binomial value is about 0.0284.) **Answer: about 0.029**

3. Central limit theorem (CLT). For a large sample from **any** population with mean μ and finite variance σ^2, the sample mean $\bar{x}$ is approximately $N(\mu, \sigma^2/n)$. This is why large-sample (z) tests use the normal table, whatever the shape of the population.

4. Uses of the normal distribution.

- Many natural measurements (heights, errors of measurement) are close to normal. Gauss used it for errors, so it is also the **normal law of errors**.
- It approximates the binomial, Poisson, chi-square and t distributions for large n.
- Sampling distributions of means and proportions are normal for large samples (CLT), so it is the basis of **large-sample tests** and confidence limits ($\bar{x} \pm 1.96$ SE for 95%).
- **Quality control:** control limits at $\mu \pm 3\sigma$; a point outside means a cause to investigate (only 0.27% fall outside by chance).
- Standardising and grading scores (Z scores).

5. Other properties.

- The total area is 1, and the curve is symmetric, so $P(Z > 0) = 0.5$.
- All odd central moments are 0; $\mu_2 = \sigma^2$, $\mu_4 = 3\sigma^4$. In general $\mu_{2r} = 1 \cdot 3 \cdot 5 \ldots (2r - 1)\, \sigma^{2r}$.
- A linear function of independent normal variables is normal.
- Changing μ shifts the curve sideways. A larger σ makes the curve flatter and wider, with a lower peak.

Fact	Value
Area $\mu \pm 0.6745\sigma$	50% (quartiles)
Area $\mu \pm 1.645\sigma$	90%
Area $\mu \pm 1.96\sigma$	95%
Area $\mu \pm 2.58\sigma$	99%

COMMON TRAPS

- ✗ "N(50, 100) has SD 100." ✓ The second parameter is the **variance**; SD = **10**.
- ✗ "P(X = 60) for a normal X is the height of the curve at 60." ✓ For any continuous variable P(X = 60) = **0**.
- ✗ "P(Z > 2) = 0.4772." ✓ The table gives area 0 to 2; P(Z > 2) = 0.5 − 0.4772 = **0.0228**.
- ✗ "Points of inflexion at μ ± 2σ." ✓ At **μ ± σ**.
- ✗ "For independent normals, V(X − Y) = $\sigma_1^2 - \sigma_2^2$." ✓ **$\sigma_1^2 + \sigma_2^2$**.
- ✗ "95% of the area lies within μ ± 2σ exactly." ✓ μ ± 2σ holds **95.45%**; 95% needs μ ± **1.96σ**.
- ✗ "QD ≈ 4σ/5 and MD ≈ 2σ/3." ✓ **QD ≈ 2σ/3, MD ≈ 4σ/5**.
- ✗ "Normal approximation for P(X ≥ 60) uses 60.5." ✓ Continuity correction moves **outward**: use **59.5**.
- ✗ "The CLT says the population becomes normal." ✓ It is the **sample mean** that becomes approximately normal.

Quick revision

READ THIS THE DAY BEFORE THE EXAM

1.1 Random variables, probability mass and density functions, mathematical expectation and variance

- pmf: $p(x) \geq 0$, $\Sigma\, p(x) = 1$; pdf: $f(x) \geq 0$, $\int f = 1$.
- Continuous: $P(X = a) = 0$; $f = F'$.
- $F(x) = P(X \leq x)$: non-decreasing, 0 to 1.
- $E(X) = \Sigma\, x\, p(x)$ or $\int x\, f(x)\, dx$.
- $V(X) = E(X^2) - [E(X)]^2$.
- $E(aX + b) = aE(X) + b$; $V(aX + b) = a^2V(X)$.
- $E(X + Y) = E(X) + E(Y)$ always; $E(XY) = E(X)E(Y)$ if independent.
- Independent: $V(X \pm Y) = V(X) + V(Y)$.
- Cov = 0 does not imply independence.
- Chebyshev: $P(|X - \mu| < k\sigma) \geq 1 - 1/k^2$.

1.2 Bernoulli, binomial and Poisson distributions

- Bernoulli: mean p, variance pq.
- $B(n, p)$: ${}^nC_x\, p^x\, q^{n-x}$; mean np; variance npq < mean.
- Binomial symmetric only at $p = 1/2$; max variance n/4.
- β_1 (binomial) $= (q - p)^2/npq$; $\beta_2 = 3 + (1 - 6pq)/npq$.
- $P(\lambda)$: e^(−λ)λ^x/x!; mean = variance = λ.
- Poisson: $\beta_1 = 1/\lambda$, $\beta_2 = 3 + 1/\lambda$; always positive skew.
- Poisson = binomial with $n \to \infty$, $p \to 0$, $np = \lambda$.
- Mode: integral part of $(n + 1)p$ or λ; bimodal if integer.
- Recurrence: binomial $(n - x)/(x + 1) \cdot p/q$; Poisson $\lambda/(x + 1)$.
- Additive: $B(n_1 + n_2, p)$; $P(\lambda_1 + \lambda_2)$.

1.3 Normal distribution: properties, standard normal, area rules and uses

- $X \sim N(\mu, \sigma^2)$; second parameter = variance.
- $Z = (X - \mu)/\sigma \sim N(0, 1)$; $\Phi(-z) = 1 - \Phi(z)$.
- Mean = median = mode; inflexion at $\mu \pm \sigma$; asymptotic.
- 68.27%, 95.45%, 99.73% for 1σ, 2σ, 3σ.
- $1.645\sigma \rightarrow 90\%$, $1.96\sigma \rightarrow 95\%$, $2.58\sigma \rightarrow 99\%$.
- $QD \approx 2\sigma/3$, $MD \approx 4\sigma/5$; $QD : MD : SD \approx 10 : 12 : 15$.
- Independent normals: $X \pm Y \sim N(\mu_1 \pm \mu_2, \sigma_1^2 + \sigma_2^2)$.
- $\bar{x} \sim N(\mu, \sigma^2/n)$; $SE = \sigma/\sqrt{n}$.
- $B(n, p) \approx N(np, npq)$; $P(\lambda) \approx N(\lambda, \lambda)$; use ± 0.5 correction.
- Uses: large-sample tests, CLT, control limits $\mu \pm 3\sigma$.

Practice questions

15 QUESTIONS · ANSWERS AT THE END

1. Which of the following pairs of a random variable and its type is NOT correctly matched?

a. Number of goals scored in a football match : Continuous

b. Number of students absent in a class : Discrete

c. Height of a student : Continuous

d. Waiting time at a bus stop : Continuous

2. $X \sim B(5, 1/2)$. Find $P(X \leq 1)$.

a. 1/32

b. 5/32

c. 3/16

d. 13/16

3. Assertion (A): The normal approximation to the binomial works well when n is large and p is close to 0.5.

Reason (R): For a normal distribution, the mean, median and mode coincide.

a. Both A and R are true, and R is the correct explanation of A

b. Both A and R are true, but R is not the correct explanation of A

c. A is true, but R is false

d. A is false, but R is true

4. Match List I with List II:

List I:

A. Probability mass function

B. Probability density function

C. Distribution function

D. Mathematical expectation of X

List II:

1. $\Sigma x\, p(x)$

2. $P(X \leq x)$

3. Point probabilities of a discrete variable that add to 1

4. Area under its curve over an interval gives a probability

Choose the correct answer:

a. A-3, B-4, C-2, D-1

b. A-4, B-3, C-2, D-1

c. A-3, B-4, C-1, D-2

d. A-3, B-2, C-4, D-1

5. $X \sim B(6, p)$ and $9\,P(X = 4) = P(X = 2)$. Find p.

a. 1/10 b. 1/4

c. 1/3 d. 3/4

6. Kiran scores 70 in a test with mean 60 and SD 5. Ravi scores 80 in another test with mean 65 and SD 10. Both tests have normal scores. Who did relatively better?

a. Ravi, because 80 is more than 70 b. Kiran, with z = 2 against Ravi's 1.5

c. Ravi, with z = 2 against Kiran's 1.5 d. Both equally, since each is above the mean

7. Consider the following statements about a continuous random variable X with pdf f(x):

1. f(x) can be greater than 1 for some values of x.

2. $P(a < X < b) = P(a \le X \le b)$.

Which of the above is/are correct?

a. 1 only b. Both 1 and 2

c. 2 only d. Neither 1 nor 2

8. $X \sim B(6, 1/3)$. Using the recurrence relation, $P(X = 3) = k \times P(X = 2)$. Find k.

a. 1/2 b. 2/3

c. 4/3 d. 3/2

9. Consider the following statements:

I. If X is normal with mean μ and variance σ^2, then for constants $a \neq 0$ and b, $aX + b$ is normal with mean $a\mu + b$ and variance $a^2\sigma^2$.

II. If X is normal with mean 50 and variance 16, then 2X has the same distribution as X1 + X2, where X1 and X2 are independent, each normal with mean 50 and variance 16.

Which of the statements given above is/are correct?

a. Only I is correct b. Only II is correct

c. Both I and II are correct d. Neither I nor II is correct

10. $f(x) = kx(1 - x)$ for $0 \le x \le 1$ (and 0 elsewhere) is a pdf. Find k.

a. 1/6 b. 2

c. 3 d. 6

11. A test has 8 questions, each with 4 options of which one is correct. A student guesses every answer. What is the expected number of correct answers?

a. 1.5 b. 2

c. 4 d. 6

12. For a normal distribution, consider:

1. The area between μ and μ + σ equals the area between μ + σ and μ + 2σ.
2. The total area under the normal curve is 1.

Which of the statements given above is/are correct?

a. 1 only b. 2 only

c. Both 1 and 2 d. Neither 1 nor 2

13. For a random variable X, $E(X) = 2$ and $E(X^2) = 8$. Find $Var(2X + 1)$.

a. 4 b. 8

c. 16 d. 17

14. Match List I with List II:

List I:

A. Mean of a Bernoulli distribution

B. Variance of a Bernoulli distribution

C. Mean of a binomial distribution

D. Variance of a binomial distribution

List II:

1. npq
2. p
3. np
4. pq

Choose the correct answer:

a. A-2, B-4, C-1, D-3 b. A-4, B-2, C-3, D-1

c. A-3, B-1, C-2, D-4 d. A-2, B-4, C-3, D-1

15. A population has mean 50 and SD 10. For a random sample of 25, find $P(\bar{x} > 52)$. (Area from 0 to z: $z = 0.2 \rightarrow 0.0793$, $z = 1 \rightarrow 0.3413$)

a. 0.1587 b. 0.3413

c. 0.4207 d. 0.8413

Answers: 1-(a) 2-(c) 3-(b) 4-(a) 5-(b) 6-(b) 7-(b) 8-(b) 9-(a) 10-(d) 11-(b) 12-(b) 13-(c) 14-(d) 15-(a)